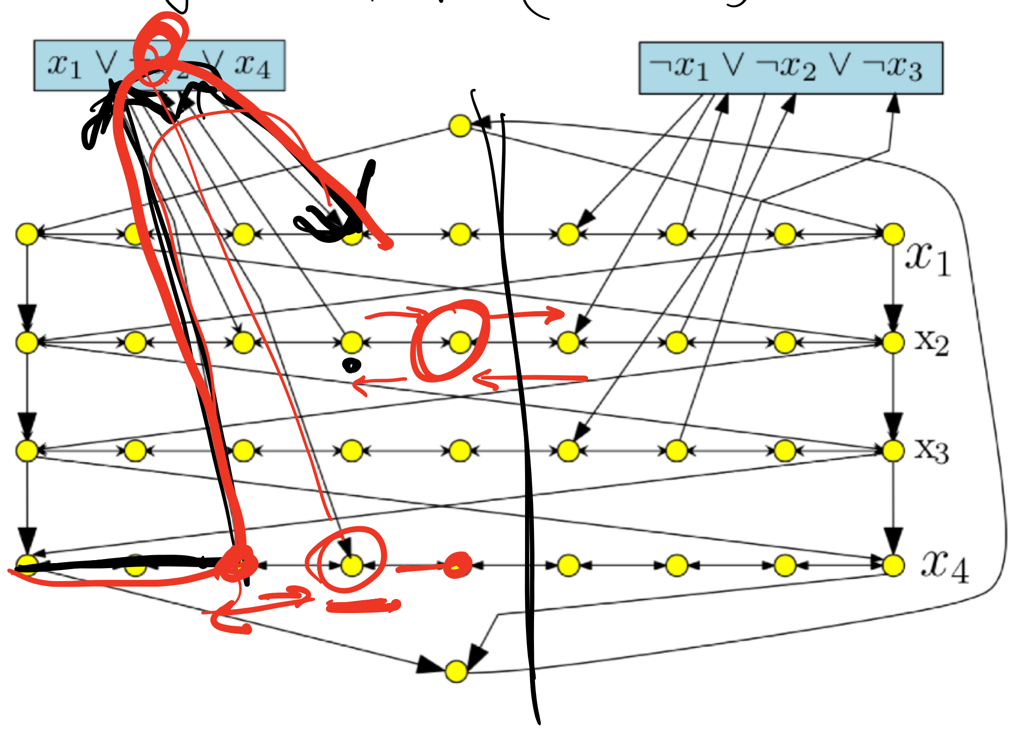
LECTURE 24 MORE NP HARDNESS.

Hamiltonian Cycle wrap-up: (No Funny Business Claim)



3SAT $\leq_p$ Directed Hamiltonian Cycle

$\leq_p$

X = Undirected Hamiltonian Cycle

To prove X is NP-hard.

Pick a known NP-hard problem (DHC)

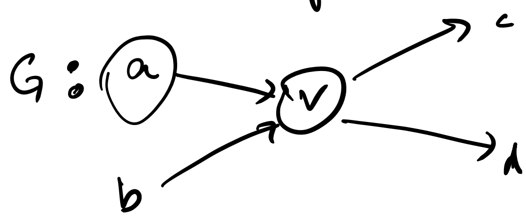
Reduce from known problem to X.

DHC $\leq_p$ UHC

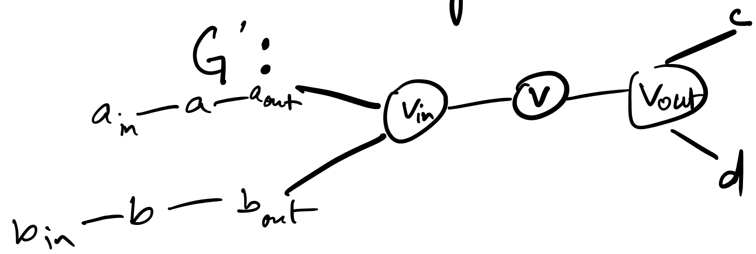
Input : Directed Graph.

Output : Undirected Graph.

Directed graph



Undirected graph



Note : paths in G' look like .... $\underline{a_{out} - V_{in}} - V - \underline{V_{out} - C_{in}}$

G has H.C. $\Rightarrow$ there is a cycle ....a $\rightarrow$ v $\rightarrow$ d.... in G

$\Rightarrow$ there is a cycle $a_{in} - a - a_{out} - V_{in} - V - V_{out}$
$d_{in} - d - d_{out}$ ...
in G'

G' has H.C. $\Rightarrow$

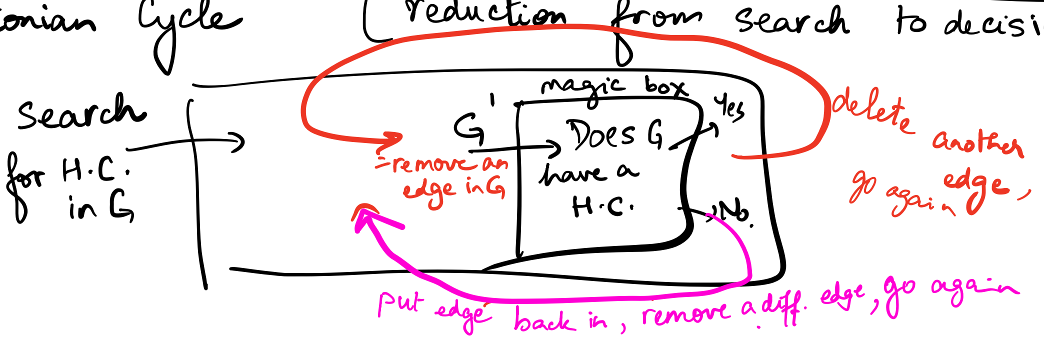
there is a cycle

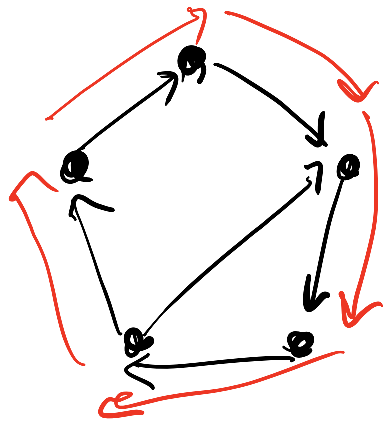
... $a_{in} - a - a_{out} - V_{in} - V - V_{out} - d_{in} - d - d_{out}$ ....
in G'.

$\Rightarrow$ a $\rightarrow$ v $\rightarrow$ d is a directed cycle in G that visits every vertex

$\Rightarrow$ a $\rightarrow$ v $\rightarrow$ d is a Hamiltonian cycle in G

Hamiltonian Cycle (reduction from search to decision)

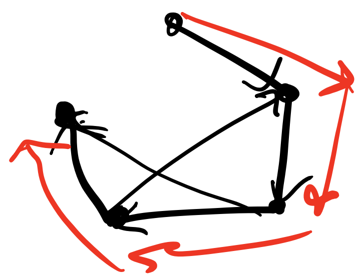




Input: An arbitrary $\boxed{G}$

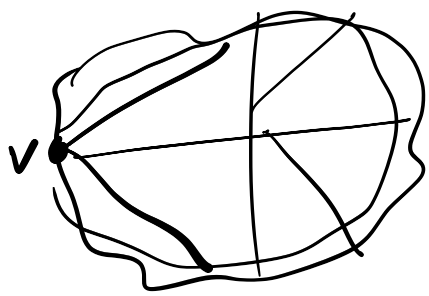
can do: use magic box on any graph you like

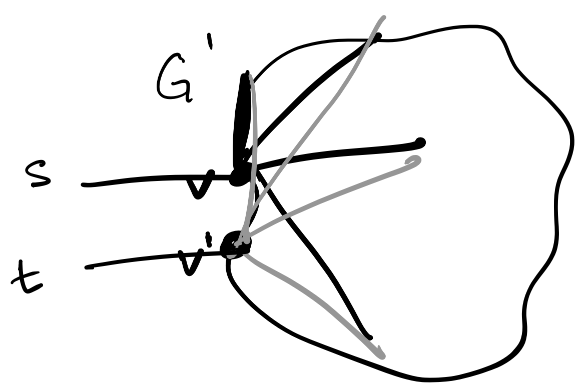
Hamiltonian Path (also NP-hard).



U·HC $\leq_p$ U H.P.
(known NP-hard)

G





G' has H.P. iff G has H.C.

Input: G.

Construct G' as follows. Pick $v \in V$ of G.

G has a H.C. $\Rightarrow$ G' has a H.P.

$v \to u \to \ldots \to w \to v$ $\quad$ $s \to v \to u \ldots \to w \to v' \to t$

G' has a H.P.

It has to be $s \to v \to u \ldots w \to v' \to t$

$\Rightarrow v \to u \ldots \to w \to v$ is a cycle in G that visits every vertex

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To prove X is NP-hard.

Pick a known NP-hard problem.

Reduce from known problem to X.

How to pick an NP-hard problem?

Suppose your problem X asks you to:

- in set of objects assign T/F subject to some constraints
  binary choices

Reduce from:
CSAT
$\boxed{3SAT}$
if nothing else works

- inset of objects assign 3 possible values or 5 values subject to constraints

3COL
kCOL

- order objects / find a long sequence

Ham. cycle/path

Traveling Salesman Problem

- Largest possible subset
- Smallest possible subset

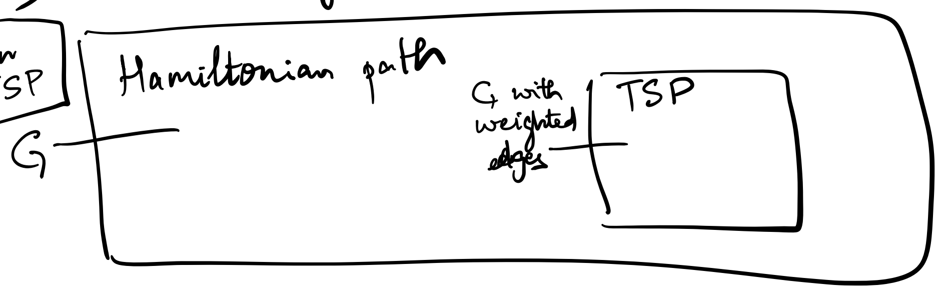
MaxClique
Max Independent set
Min Vertex Cover

Other NP-hard problems.

TSP (is NP-hard)

Length of shortest Hamiltonian path / cycle in a graph with weighted edges, if it exists,

$\boxed{\text{Hamiltonian path} \le_p \text{TSP}}$



SUBSETSUM.

Given array of positive integers, and positive integer k,

- is there a subset of integers that sums to k?

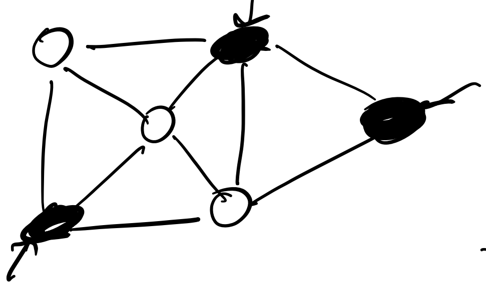
PARTITION

Given an array of positive integers,

- Divide elements of array into two sets S_1, S_2 such that they don't intersect and such that sum of elements in S_1 = sum of elements in S_2.

EXAMPLE SOLVED QUESTIONS.

1. A subset S of vertices in an undirected graph G is *half-independent* if each vertex in S is adjacent to *at most one* other vertex in S. Prove that finding the size of the largest half-independent set of vertices in a given undirected graph is NP-hard.

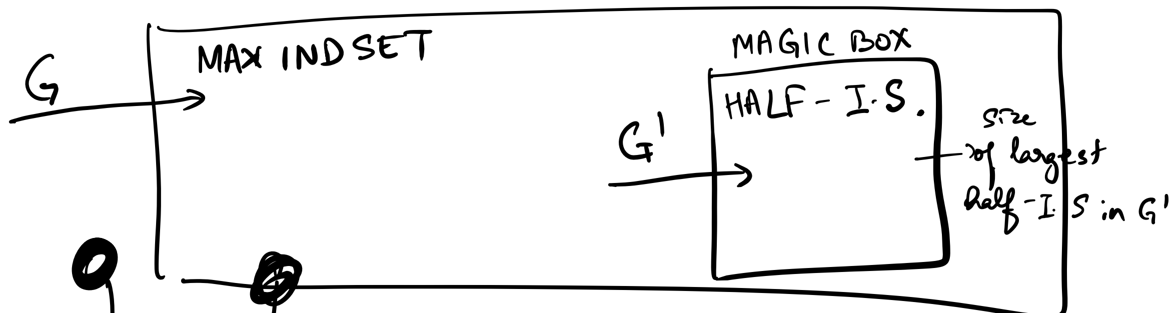


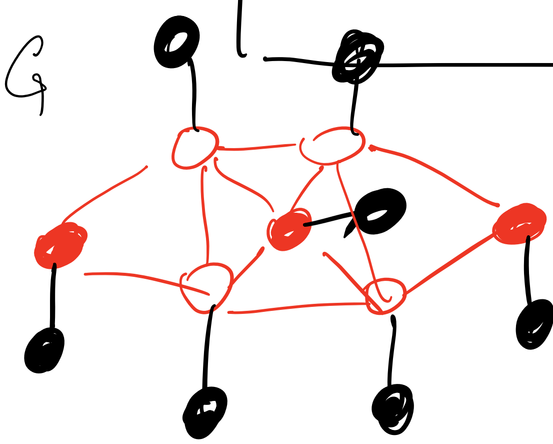
Independent Set problem

X is NP-hard

Known problem $\leq_p X$

from INDEPENDENT SET $\leq_p$ HALF-IND SET
to





G'

G' adds a new vertex to every vertex in G.
Call these new vertices L.

Let S be any independent set in G.

$\Rightarrow S \cup L$ is half independent in G'.

Every vertex in S is connected just to 1 other vertex (its clone)

Every vertex in L is connected to atmost 1 other vertex.

$\Rightarrow$ $S \cup L$ is half independent in G'.

Suppose X is a half-independent set in G'.

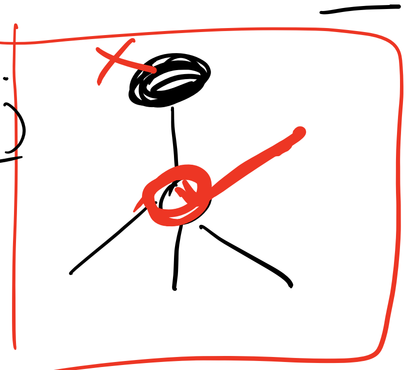
then remove L from X?

$\Rightarrow$ $X \setminus L = S$ is an independent set in G.

Sub-Claim : Some largest Half ind set in G' contains L.

Suppose not. Then there is a largest half ind set that doesn't contain some vertex. Two cases:

(1)



change the set to contain black and remove red

or (2)

include black in the set

Therefore, we obtain a largest half ind. set X in G' that contains every vertex in L.

Remove L from X to obtain S.

Easy exercise to prove that S is an independent set in G.

2 A subset S of vertices in an undirected graph G is *sort-of-independent* if if each vertex in S is adjacent to *at most 374* other vertices in S. Prove that finding the size of the largest sort-of-independent set of vertices in a given undirected graph is NP-hard.

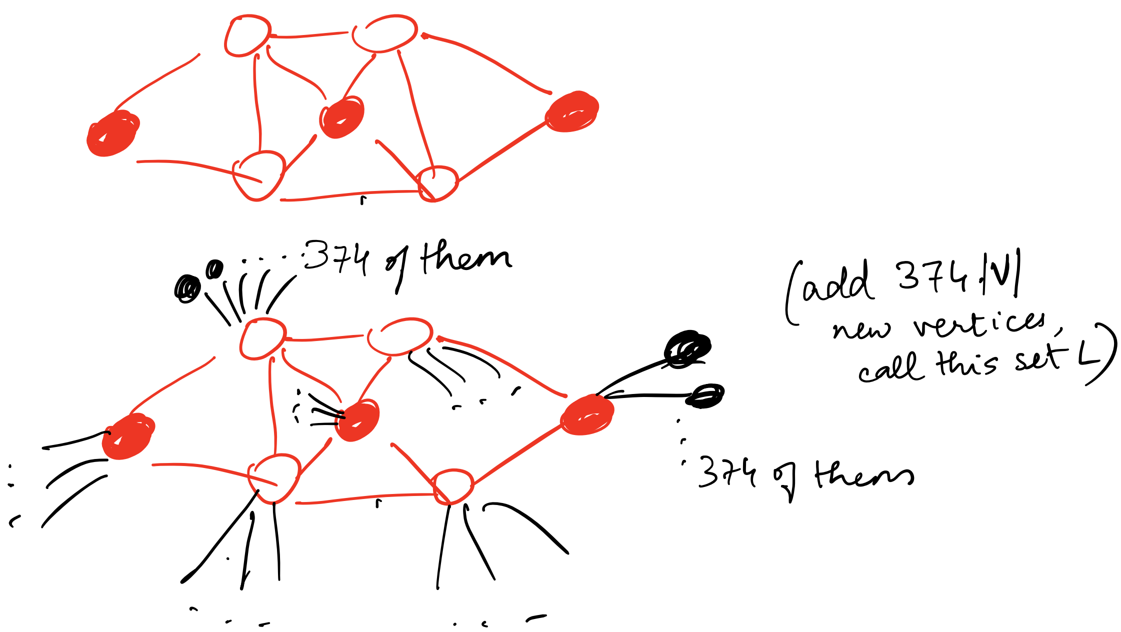
Left as an exercise

Hint :

G

$\Downarrow$

G'



Try to prove that :

G has a (largest) independent set S

$\Updownarrow$

G' has a (largest) independent set $S \cup L$

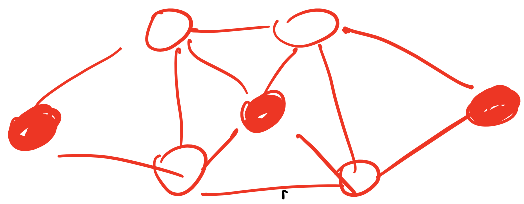
3

A subset S of vertices in an undirected graph G is *almost independent* if at most 374 edges in G have both endpoints in S. Prove that finding the size of the largest almost-independent set of vertices in a given undirected graph is NP-hard.

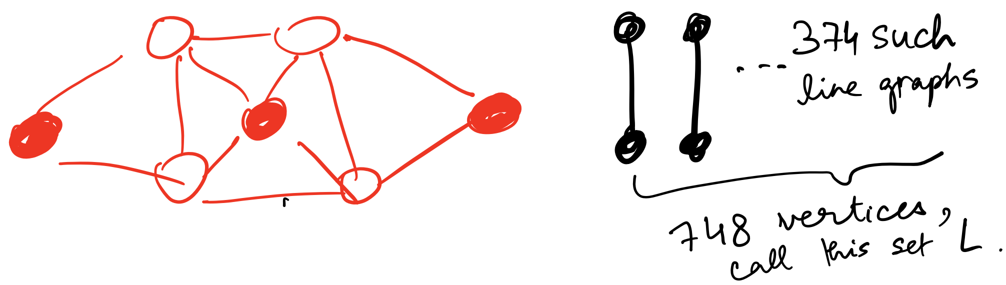
Left as an exercise

Hint:

G:



G':



Try to prove that:

G has an independent set S

$\iff$ G' has an almost independent set $S \cup L$