HWII will be out later today (due 2 weeks from now).
GPSII will be out later today.

\( \mathbf{P} \) - solvable in polynomial time

\( \mathbf{NP} \) - checkable in polynomial time.

\( \mathbf{X} \) is \( \mathbf{NP} \)-hard - If poly time algo for \( \mathbf{X} \),
\[ \begin{array}{c}
\text{existence of this} \\
\text{is also really unlikely}
\end{array} \]
then \( \mathbf{P} = \mathbf{NP} \) really unlikely.

To prove \( \mathbf{X} \) is \( \mathbf{NP} \)-hard:

Pick a \underline{known} \( \mathbf{NP} \)-hard problem

\((\text{Cook-Levin}: \text{Circuit-SAT is NP-hard})\)

\( \text{CSAT} \leq_p \text{3SAT} \) :: \( \text{3SAT is NP-hard} \)

\( \text{3SAT} \leq_p \text{MaxClique} \) :: \( \text{MaxClique is NP-hard} \)

\( \text{3SAT} \leq_p \text{MaxIndSet} \) :: \( \text{IndepSet is NP-hard} \)

\( \text{MaxIndSet} \leq_p \text{MinVC} \) :: \( \text{MinVerCov is NP-hard} \)
\[
\text{MAXINDEPSET} \leq_p \text{MAXCLIQUE}
\]

\[
\text{MAXINDEPSET} \leq_p \text{MINVERTEXCOVER}
\]
2SAT \leq_p 3SAT

(a \lor b) \rightarrow (a \lor b \lor c) \land (a \lor b \lor \neg c)

2SAT is easy.
3SAT is NP-hard.

3SAT \leq_p 2SAT ? Likely not true
(If true, then P=NP)

\textbf{3 color}

\textbf{Input:} G = (V, E)

\textbf{Problem:} Can we color vertices red, green, blue such that every edge touches two colors?
(no 2 vertices connected by an edge share the same color).

\textbf{Claim:} 3-color is NP-hard.

\textbf{Proof:} 3SAT \leq_p 3-color
Formula: \((a \lor b \lor c) \land (b \lor c \lor \overline{d})\)

Graph:

Truth gadget

Variable gadget (one for each variable)

Clause gadget: \((a \lor b \lor c)\)
Claim 1: Clause gadget has a 3-coloring \( \iff \) there is at least 1 true literal in the clause

Proof by picture that if \( a, b, c \) are all false \( \Rightarrow \) no 3COL exists
(by negation) If 3COL exists \( \Rightarrow \) 1 of \( a, b, c \) must be true

If 1 of \( a, b, c \) is true \( \Rightarrow \) 3 col exists.

Claim 1 \( \Rightarrow \)
Every clause must have at least 1 true literal if and only if \( G \) has a 3-COL

POLYNOMIAL TIME: \( \therefore 2\text{SAT} \leq \text{P 3-COL} \)
3-COL \( \leq \text{P 4-COL} \leq \text{P 5-COL} \)
**Hamiltonian Cycle**

Given graph $G = (V, E)$

Is there a cycle that touches every vertex exactly once?

**Claim:** Hamiltonian cycle is NP-complete.

**Proof:** Hamiltonian cycle is in NP, (exercise)

Hamiltonian cycle is NP-hard.

$3\text{SAT} \leq_p \text{Ham cycle in directed graphs}$

Given 3-CNF formula $\varphi \Rightarrow$ convert to $G$, s.t. $G$ has Ham. cycle $\Leftrightarrow \varphi$ is satisfiable.

Encoded variables into the graph

3-CNF: formula: $(x_1 \lor x_2 \lor x_3) \land (x_1 \lor x_2 \land x_4)$ $2^n$ Hamiltonian cycles
Encode clause into the graph

3k+3 vertices in each row, where k is number of clauses.

Add vertex $c_j$ for every clause $C_j$.
$c_j$ has edge from $3j$ and to $3j+1$ on $i^{th}$ row if $x_i \in C_j$.
$c_j$ has edge from $3j+1$ and to $3j$ on $i^{th}$ row if $\neg x_i \in C_j$.
Claim: There is a SAT assignment to the 3CNF formula $\iff$ there is a Hamiltonian cycle on the graph.

Proof:

Suppose there is a SAT assignment.

$\Rightarrow$ there is a Ham. cycle.

Suppose there is a Ham. cycle

$\Rightarrow$ SAT assignment.
Ham. cycle can make at most 1-hop detours on G. Traverse each row in a single direction.