Consider the following recursively defined function on strings:

\[ \text{stutter}(w) := \begin{cases} 
\epsilon & \text{if } w = \epsilon \\
\alpha \cdot \text{stutter}(x) & \text{if } w = \alpha x \text{ for some symbol } \alpha \text{ and some string } x
\end{cases} \]

Intuitively, \( \text{stutter}(w) \) doubles every symbol in \( w \). For example:

- \( \text{stutter}(\text{PRESTO}) = \text{PPREESSTTOO} \)
- \( \text{stutter}(\text{HOCUS} \circ \text{POCUS}) = \text{HHOOCUUSS} \circ \text{POOCCUUSS} \)

Let \( L \) be an arbitrary regular language.

1. Prove that the language \( \text{Unstutter}(L) := \{ w \mid \text{stutter}(w) \in L \} \) is regular.
2. Prove that the language \( \text{Stutter}(L) := \{ \text{stutter}(w) \mid w \in L \} \) is regular.

Work on these later:

3. Let \( L \) be an arbitrary regular language.
   
   (a) Prove that the language \( \text{Insert}^1(L) := \{ x \, y \mid x \, y \in L \} \) is regular.
   
   Intuitively, \( \text{Insert}^1(L) \) is the set of all strings that can be obtained from strings in \( L \) by inserting exactly one \( 1 \). For example:
   
   \[ \text{Insert}^1(\{\epsilon, \, 00, \, 101101\}) = \{1, \, 100, \, 010, \, 001, \, 1101101, \, 1011011, \, 1011011\} \]

   (b) Prove that the language \( \text{Delete}^1(L) := \{ x \, y \mid x \, 1 \, y \in L \} \) is regular.
   
   Intuitively, \( \text{Delete}^1(L) \) is the set of all strings that can be obtained from strings in \( L \) by deleting exactly one \( 1 \). For example:
   
   \[ \text{Delete}^1(\{\epsilon, \, 00, \, 101101\}) = \{01101, \, 10101, \, 10110\} \]

4. Consider the following recursively defined function on strings:

\[ \text{evens}(w) := \begin{cases} 
\epsilon & \text{if } w = \epsilon \\
\epsilon & \text{if } w = a \text{ for some symbol } a \\
b \cdot \text{evens}(x) & \text{if } w = abx \text{ for some symbols } a \text{ and } b \text{ and some string } x
\end{cases} \]

Intuitively, \( \text{evens}(w) \) skips over every other symbol in \( w \). For example:

- \( \text{evens}(\text{EXPPELLIARMUS}) = \text{XELAMS} \)
- \( \text{evens}(\text{AVADA} \circ \text{KEDAVRA}) = \text{VD} \circ \text{EAR} \).

Once again, let \( L \) be an arbitrary regular language.

(a) Prove that the language \( \text{Unevens}(L) := \{ w \mid \text{evens}(w) \in L \} \) is regular.
(b) Prove that the language \( \text{Evens}(L) := \{ \text{evens}(w) \mid w \in L \} \) is regular.