Proving that a problem $X$ is NP-hard requires several steps:

- Choose a problem $Y$ that you already know is NP-hard (because we told you so in class).
- Describe an algorithm to solve $Y$, using an algorithm for $X$ as a subroutine. Typically this algorithm has the following form: Given an instance of $Y$, transform it into an instance of $X$, and then call the magic black-box algorithm for $X$.
- **Prove** that your algorithm is correct. This always requires two separate steps, which are usually of the following form:
  - **Prove** that your algorithm transforms “good” instances of $Y$ into “good” instances of $X$.
  - **Prove** that your algorithm transforms “bad” instances of $Y$ into “bad” instances of $X$. Equivalently: Prove that if your transformation produces a “good” instance of $X$, then it was given a “good” instance of $Y$.
- Argue that your algorithm for $Y$ runs in polynomial time. (This is usually trivial.)

1. Suppose you are given a magic black box that somehow answers the following decision problem in polynomial time:

   - **INPUT**: A boolean circuit $K$ with $n$ inputs and one output.
   - **OUTPUT**: TRUE if there are input values $x_1, x_2, \ldots, x_n \in \{\text{TRUE}, \text{FALSE}\}$ that make $K$ output TRUE, and FALSE otherwise.

   Using this black box as a subroutine, describe an algorithm that solves the following related search problem in polynomial time:

   - **INPUT**: A boolean circuit $K$ with $n$ inputs and one output.
   - **OUTPUT**: Input values $x_1, x_2, \ldots, x_n \in \{\text{TRUE}, \text{FALSE}\}$ that make $K$ output TRUE, or NONE if there are no such inputs.

   *[Hint: You can use the magic box more than once.]*

2. A Hamiltonian cycle in a graph $G$ is a cycle that goes through every vertex of $G$ exactly once. Deciding whether an arbitrary graph contains a Hamiltonian cycle is NP-hard.

   A tonian cycle in a graph $G$ is a cycle that goes through at least half of the vertices of $G$. Prove that deciding whether a graph contains a tonian cycle is NP-hard.

**To think about later:**

3. Let $G$ be an undirected graph with weighted edges. A Hamiltonian cycle in $G$ is heavy if the total weight of edges in the cycle is at least half of the total weight of all edges in $G$. Prove that deciding whether a graph contains a heavy Hamiltonian cycle is NP-hard.
A heavy Hamiltonian cycle. The cycle has total weight 34; the graph has total weight 67.