1. Prove that the following languages are not regular.

   (a) \( \{ \theta^m 1^n \mid m \text{ and } n \text{ are relatively prime} \} \)

   (b) \( \{ w \in (0 + 1)^* \mid 10^n 1^n \text{ for } n > 0 \text{ is a suffix of } w \} \)

   (c) The set of all palindromes in \((0 + 1)^*\) whose length is divisible by 3.

2. For each of the following languages over the alphabet \( \Sigma = \{0, 1\} \), either prove that the language is regular (by constructing an appropriate DFA, NFA, or regular expression) or prove that the language is not regular (by constructing an infinite fooling set). Recall that \( \Sigma^+ \) denotes the set of all nonempty strings over \( \Sigma \). Watch those parentheses!

   (a) \( \{ \theta^a 1w10^c \mid w \in \Sigma^+, (a \leq |w| + c) \text{ and } (|w| \leq a + c \text{ or } c \leq a + |w|) \} \)

   (b) \( \{ \theta^a w \theta^a \mid w \in \Sigma^+, a > 0, |w| \geq 0 \} \)

   (c) \( \{ xww^Ry \mid w, x, y \in \Sigma^+ \} \)

   (d) \( \{ w^Rxy \mid w, x, y \in \Sigma^+ \} \)

   [Hint: Exactly two of these languages are regular.]
Solved problem

4. For each of the following languages, either prove that the language is regular (by constructing an appropriate DFA, NFA, or regular expression) or prove that the language is not regular (by constructing an infinite fooling set).

Recall that a palindrome is a string that equals its own reversal: \( w = w^R \). Every string of length 0 or 1 is a palindrome.

(a) Strings in \((\{0,1\}\}^*\) in which no prefix of length at least 2 is a palindrome.

\[ \text{Solution: Regular: } \varepsilon + 01^* + 10^*. \text{ Call this language } L_a. \]

Let \( w \) be an arbitrary non-empty string in \((\{0,1\}\}^*\). Without loss of generality, assume \( w = \varepsilon x \) for some string \( x \). There are two cases to consider.

- If \( x \) contains a \( 0 \), then we can write \( w = 01^n y \) for some integer \( n \) and some string \( y \). The prefix \( 01^n \) is a palindrome of length at least 2. Thus, \( w \notin L_a \).
- Otherwise, \( x \in 1^* \). Every non-empty prefix of \( w \) is equal to \( 01^n \) for some non-negative integer \( n \leq |x| \). Every palindrome that starts with \( 0 \) also ends with \( 0 \), so the only palindrome prefixes of \( w \) are \( \varepsilon \) and \( 0 \), both of which have length less than 2. Thus, \( w \in L_a \).

We conclude that \( \varepsilon x \in L_a \) if and only if \( x \in 1^* \). A similar argument implies that \( 1x \in L_a \) if and only if \( x \in 0^* \). Finally, trivially, \( \varepsilon \in L_a \). \( \blacksquare \)

\[ \text{Rubric: } 2\frac{1}{2} \text{ points = } \frac{1}{2} \text{ for “regular” + 1 for regular expression + 1 for justification. This is more detail than necessary for full credit.} \]

(b) Strings in \((\{0,1,2\}\}^*\) in which no prefix of length at least 2 is a palindrome.

\[ \text{Solution: Not regular. Call this language } L_b. \]

I claim that the infinite language \( F = (012)^+ \) is a fooling set for \( L_b \).

Let \( x \) and \( y \) be arbitrary distinct strings in \( F \).

Then \( x = (012)^i \) and \( y = (012)^j \) for some positive integers \( i \neq j \).

Without loss of generality, assume \( i < j \).

Let \( z \) be the suffix \( (210)^i \).

- \( xz = (012)^i (210)^i \) is a palindrome of length \( 6i \geq 2 \), so \( xz \notin L_b \).
- \( yz = (012)^i (210)^i \) has no palindrome prefixes except \( \varepsilon \) and \( 0 \), because \( i < j \), so \( yz \in L_b \).

We conclude that \( F \) is a fooling set for \( L_b \), as claimed.

Because \( F \) is infinite, \( L_b \) cannot be regular. \( \blacksquare \)

\[ \text{Rubric: } 2\frac{1}{2} \text{ points = } \frac{1}{2} \text{ for “not regular” + 2 for fooling set proof (standard rubric, scaled).} \]
(c) Strings in \((0 + 1)^*\) in which no prefix of length at least 3 is a palindrome.

**Solution:** Not regular. Call this language \(L_c\).

I claim that the infinite language \(F = (001101)^+\) is a fooling set for \(L_c\).

Let \(x\) and \(y\) be arbitrary distinct strings in \(F\).

Then \(x = (001101)^i\) and \(y = (001101)^j\) for some positive integers \(i \neq j\).

Without loss of generality, assume \(i < j\).

Let \(z\) be the suffix \((101100)^i\).

- \(xz = (001101)^i(101100)^i\) is a palindrome of length \(12i \geq 2\), so \(xz \notin L_b\).
- \(yz = (001101)^i(101100)^i\) has no palindrome prefixes except \(\epsilon\) and \(0\), because \(i < j\), so \(yz \in L_b\).

We conclude that \(F\) is a fooling set for \(L_c\), as claimed.

Because \(F\) is infinite, \(L_c\) cannot be regular. \(\blacksquare\)

**Rubric:** 2½ points = ½ for “not regular” + 2 for fooling set proof (standard rubric, scaled).

(d) Strings in \((0 + 1)^*\) in which no substring of length at least 3 is a palindrome.

**Solution:** Regular. Call this language \(L_d\).

Every palindrome of length at least 3 contains a palindrome substring of length 3 or 4. Thus, the complement language \(\overline{L_d}\) is described by the regular expression

\[
(0 + 1)^*(000 + 010 + 011 + 111 + 0110 + 1001)(0 + 1)^*
\]

Thus, \(\overline{L_d}\) is regular, so its complement \(L_d\) is also regular. \(\blacksquare\)

**Solution:** Regular. Call this language \(L_d\).

In fact, \(L_d\) is finite! Appending either 0 or 1 to any of the underlined strings creates a palindrome suffix of length 3 or 4.

\[
\epsilon + 0 + 1 + 00 + 01 + 10 + 11 + 001 + 011 + 100 + 110 + 0011 + 1100
\]

**Rubric:** 2½ points = ½ for “regular” + 2 for proof:

- 1 for expression for \(\overline{L_d}\) + 1 for applying closure
- 1 for regular expression + 1 for justification
**Standard fooling set rubric.** For problems worth 5 points:

- 2 points for the fooling set:
  - + 1 for explicitly describing the proposed fooling set $F$.
  - + 1 if the proposed set $F$ is actually a fooling set for the target language.
  - − No credit for the proof if the proposed set is not a fooling set.
  - − No credit for the problem if the proposed set is finite.

- 3 points for the proof:
  - ◦ The proof must correctly consider arbitrary strings $x, y \in F$.
    - − No credit for the proof unless both $x$ and $y$ are always in $F$.
    - − No credit for the proof unless $x$ and $y$ can be any strings in $F$.
  - + 1 for correctly describing a suffix $z$ that distinguishes $x$ and $y$.
  - + 1 for proving either $xz \in L$ or $yz \in L$.
  - + 1 for proving either $yz \notin L$ or $xz \notin L$, respectively.

As usual, scale partial credit (rounded to nearest ½) for problems worth fewer points.