Introduction to Dynamic Programming

Lecture 13
Thursday, October 8, 2020
13.1
Recursion and Memoization
13.1.1
Fibonacci Numbers
Fibonacci Numbers

Fibonacci numbers defined by recurrence:

\[ F(n) = F(n - 1) + F(n - 2) \text{ and } F(0) = 0, F(1) = 1. \]

These numbers have many interesting properties. A journal The Fibonacci Quarterly!

- **Binet’s formula:**
  \[ F(n) = \frac{\varphi^n - (1 - \varphi)^n}{\sqrt{5}} \approx \frac{1.618^n - (-0.618)^n}{\sqrt{5}} \approx \frac{1.618^n}{\sqrt{5}} \]
  \[ \varphi \text{ is the golden ratio } (1 + \sqrt{5})/2 \approx 1.618. \]

- \[ \lim_{n \to \infty} F(n + 1)/F(n) = \varphi \]
Fibonacci Numbers

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  \( \varphi \) is the golden ratio \( (1 + \sqrt{5})/2 \approx 1.618. \)

- \( \lim_{n \to \infty} F(n + 1)/F(n) = \varphi \)
How many bits?

Consider the $n$th Fibonacci number $F(n)$. Writing the number $F(n)$ in base 2 requires

- $\Theta(n^2)$ bits.
- $\Theta(n)$ bits.
- $\Theta(\log n)$ bits.
- $\Theta(\log \log n)$ bits.
Recursive Algorithm for Fibonacci Numbers

**Question:** Given $n$, compute $F(n)$.

```python
Fib(n):
    if (n = 0)
        return 0
    else if (n = 1)
        return 1
    else
        return Fib(n - 1) + Fib(n - 2)
```

Running time? Let $T(n)$ be the number of additions in Fib(n).

$$T(n) = T(n - 1) + T(n - 2) + 1 \text{ and } T(0) = T(1) = 0$$
Recursive Algorithm for Fibonacci Numbers

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Recursive Algorithm for Fibonacci Numbers

Question: Given \( n \), compute \( F(n) \).

\[
\text{Fib}(n):
\begin{align*}
\text{if } (n = 0) & \quad \text{return } 0 \\
\text{else if } (n = 1) & \quad \text{return } 1 \\
\text{else} & \quad \text{return } \text{Fib}(n - 1) + \text{Fib}(n - 2)
\end{align*}
\]

Running time? Let \( T(n) \) be the number of additions in Fib\( (n) \).

\[
T(n) = T(n - 1) + T(n - 2) + 1 \quad \text{and} \quad T(0) = T(1) = 0
\]

Roughly same as \( F(n) \): \( T(n) = \Theta(\varphi^n) \).

The number of additions is exponential in \( n \). Can we do better?
Recursion tree for the Recursive Fibonacci

0  1
Recursion tree for the Recursive Fibonacci

```
    2
   / \
0    1
```

Recursion tree for the Recursive Fibonacci
Recursion tree for the Recursive Fibonacci
Recursion tree for the Recursive Fibonacci

```
3  4
  1 1  3 2
  0 1 0 1
  0 1
```

```
5
  3 4
  1 2 2 3
  0 1 0 1 1
  0 1
```
Recursion tree for the Recursive Fibonacci
Recursion tree for the Recursive Fibonacci
An iterative algorithm for Fibonacci numbers

\[
\text{FibIter}(n): \\
\text{if } (n = 0) \text{ then} \\
\quad \text{return } 0 \\
\text{if } (n = 1) \text{ then} \\
\quad \text{return } 1 \\
F[0] = 0 \\
F[1] = 1 \\
\text{for } i = 2 \text{ to } n \text{ do} \\
\quad F[i] = F[i - 1] + F[i - 2] \\
\text{return } F[n]
\]

What is the running time of the algorithm? \(O(n)\) additions.
An iterative algorithm for Fibonacci numbers

\[ \text{FibIter}(n): \]
\[
\begin{align*}
\text{if} & \ (n = 0) \ \text{then} \\
& \quad \text{return} \ 0 \\
\text{if} & \ (n = 1) \ \text{then} \\
& \quad \text{return} \ 1 \\
F[0] & = 0 \\
F[1] & = 1 \\
\text{for} & \ i = 2 \ \text{to} \ n \ \text{do} \\
& \quad F[i] = F[i - 1] + F[i - 2] \\
& \text{return} \ F[n]
\end{align*}
\]

What is the running time of the algorithm? \( O(n) \) additions.
An iterative algorithm for Fibonacci numbers

\[ \text{FibIter}(n): \]
\[
\text{if } (n = 0) \text{ then return 0}
\]
\[
\text{if } (n = 1) \text{ then return 1}
\]
\[
F[0] = 0
\]
\[
F[1] = 1
\]
\[
\text{for } i = 2 \text{ to } n \text{ do}
\]
\[
F[i] = F[i - 1] + F[i - 2]
\]
\[
\text{return } F[n]
\]

What is the running time of the algorithm? \( O(n) \) additions.
What is the difference?

1. Recursive algorithm is computing the same numbers again and again.
2. Iterative algorithm is storing computed values and building bottom up the final value. Memoization.

Dynamic Programming:
Finding a recursion that can be effectively/efficiently memoized.

Leads to polynomial time algorithm if number of sub-problems is polynomial in input size.
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What is the difference?

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**Dynamic Programming:**

Finding a recursion that can be *effectively/efficiently* memoized.

Leads to polynomial time algorithm if number of sub-problems is polynomial in input size.
THE END

... (for now)
13.1.2
Automatic/implicit memoization
Automatic Memoization

Can we convert recursive algorithm into an efficient algorithm without explicitly doing an iterative algorithm?

\[
\text{Fib}(n) =
\begin{cases} 
0 & \text{if } n = 0 \\
1 & \text{if } n = 1 \\
\text{if } (\text{Fib}(n) \text{ was previously computed}) & \text{return stored value of Fib}(n) \\
\text{else} & \text{return Fib}(n - 1) + \text{Fib}(n - 2)
\end{cases}
\]

How do we keep track of previously computed values?
Two methods: explicitly and implicitly (via data structure)
Automatic Memoization

Can we convert recursive algorithm into an efficient algorithm without explicitly doing an iterative algorithm?

\[
\text{Fib}(n): \\
\quad \text{if } (n = 0) \quad \text{return } 0 \\
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\quad \text{else} \quad \text{return } \text{Fib}(n - 1) + \text{Fib}(n - 2)
\]

How do we keep track of previously computed values?
Two methods: explicitly and implicitly (via data structure)
Automatic Memoization

Can we convert recursive algorithm into an efficient algorithm without explicitly doing an iterative algorithm?

```
Fib(n):
    if (n = 0)
        return 0
    if (n = 1)
        return 1
    if (Fib(n) was previously computed)
        return stored value of Fib(n)
    else
        return Fib(n−1) + Fib(n−2)
```

How do we keep track of previously computed values?

Two methods: explicitly and implicitly (via data structure)
Automatic Memoization

Can we convert recursive algorithm into an efficient algorithm without explicitly doing an iterative algorithm?

\[
\text{Fib}(n):
\]
\[
\begin{align*}
&\text{if } (n = 0) \\
&\quad \text{return } 0 \\
&\text{if } (n = 1) \\
&\quad \text{return } 1 \\
&\text{if } (\text{Fib}(n) \text{ was previously computed}) \\
&\quad \text{return stored value of Fib}(n) \\
&\text{else} \\
&\quad \text{return } \text{Fib}(n - 1) + \text{Fib}(n - 2)
\end{align*}
\]

How do we keep track of previously computed values?
Two methods: explicitly and implicitly (via data structure)
Automatic implicit memoization

Initialize a (dynamic) dictionary data structure $D$ to empty

$$\text{Fib}(n):$$

\[
\begin{align*}
\text{if } (n = 0) & \quad \text{return } 0 \\
\text{if } (n = 1) & \quad \text{return } 1 \\
\text{if } (n \text{ is already in } D) & \quad \text{return value stored with } n \text{ in } D \\
\text{val } \leftarrow \text{Fib}(n - 1) + \text{Fib}(n - 2) \\
\text{Store } (n, \text{val}) \text{ in } D \\
\text{return } \text{val}
\end{align*}
\]

Use hash-table or a map to remember which values were already computed.
Explicit memoization (not automatic)

1. Initialize table/array $M$ of size $n$: $M[i] = -1$ for $i = 0, \ldots, n$.

2. Resulting code:

   ```python
   def Fib(n):
       if (n == 0):
           return 0
       if (n == 1):
           return 1
       if (M[n] != -1) #: M[n]: stored value of Fib(n)
           return M[n]
       M[n] = Fib(n - 1) + Fib(n - 2)
       return M[n]
   ```

3. Need to know upfront the number of subproblems to allocate memory.
Explicit memoization (not automatic)

1. Initialize table/array $M$ of size $n$: $M[i] = -1$ for $i = 0, \ldots, n$.

2. Resulting code:

   ```python
   def Fib(n):
       if (n == 0)
           return 0
       if (n == 1)
           return 1
       if ($M[n] \neq -1$) // $M[n]$: stored value of $\text{Fib}(n)$
           return $M[n]$
       $M[n] \leftarrow \text{Fib}(n - 1) + \text{Fib}(n - 2)$
       return $M[n]$
   ```

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2. Resulting code:

   ```
   Fib(n):
   if (n = 0)
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       return 1
   if (M[n] \neq -1) // M[n]: stored value of Fib(n)
       return M[n]
   M[n] \leftarrow Fib(n - 1) + Fib(n - 2)
   return M[n]
   ```

3. Need to know upfront the number of subproblems to allocate memory.
Recursion tree for the memoized Fib...
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Automatic Memoization

Recursive version:

\[ f(x_1, x_2, \ldots, x_d) : \]

Recursive version with memoization:

\[ g(x_1, x_2, \ldots, x_d) : \]

if \( f \) already computed for \( (x_1, x_2, \ldots, x_d) \) then

\[ \text{return value already computed} \]

NEW_CODE

NEW_CODE:

1. Replaces any “return \( \alpha \)” with
2. Remember “\( f(x_1, \ldots, x_d) = \alpha \)” ; return \( \alpha \).
Automatic Memoization

1. Recursive version:

\[ f(x_1, x_2, \ldots, x_d) : \]

2. Recursive version with memoization:

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\begin{align*}
g(x_1, x_2, \ldots, x_d) : \\
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\quad \quad \text{return value already computed} \\
\end{align*}
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g(x_1, x_2, \ldots, x_d):
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\text{if } f \text{ already computed for } (x_1, x_2, \ldots, x_d) \text{ then}
\]

\[
\text{return value already computed}
\]

NEW_CODE:

1. Replaces any \texttt{return } \alpha \texttt{ with}
2. Remember \( f(x_1, \ldots, x_d) = \alpha \); \texttt{return } \alpha.
Explicit vs Implicit Memoization

 Explicit memoization (on the way to iterative algorithm) preferred:
  1. analyze problem ahead of time
  2. Allows for efficient memory allocation and access.

 Implicit (automatic) memoization:
  1. problem structure or algorithm is not well understood.
  2. Need to pay overhead of data-structure.
  3. Functional languages (e.g., LISP) automatically do memoization, usually via hashing based dictionaries.
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   3. Functional languages (e.g., LISP) automatically do memoization, usually via hashing based dictionaries.
Explicit/implicit memoization for Fibonacci

Init: $M[i] = -1, \ i = 0, \ldots, n.$

Fib$(k)$:
- if $(k = 0)$
  - return $0$
- if $(k = 1)$
  - return $1$
- if $(M[k] \neq -1)$
  - return $M[n]$

$M[k] \leftarrow \text{Fib}(k - 1) + \text{Fib}(k - 2)$

return $M[k]$

Explicit memoization

Init: Init dictionary $D$

Fib$(n)$:
- if $(n = 0)$
  - return $0$
- if $(n = 1)$
  - return $1$
- if $(n \text{ is already in } D)$
  - return value stored with $n$ in $D$

val $\leftarrow \text{Fib}(n - 1) + \text{Fib}(n - 2)$

Store $(n, val)$ in $D$

return $val$

Implicit memoization
How many distinct calls does $\text{binom}(n, \lfloor n/2 \rfloor)$ makes during its recursive execution?

- $\Theta(1)$.
- $\Theta(n)$.
- $\Theta(n \log n)$.
- $\Theta(n^2)$.
- $\Theta\left(\binom{n}{\lfloor n/2 \rfloor}\right)$.

That is, if the algorithm calls recursively $\text{binom}(17, 5)$ about 5000 times during the computation, we count this is a single distinct call.
Running time of memoized binom?

\[ D: \text{ Initially an empty dictionary.} \]
\[ \text{binomM}(t, b) \quad \text{// computes} \binom{t}{b} \]
\[ \text{if } b = t \text{ then return 1} \]
\[ \text{if } b = 0 \text{ then return 0} \]
\[ \text{if } D[t, b] \text{ is defined then return } D[t, b] \]
\[ D[t, b] \leftarrow \text{binomM}(t - 1, b - 1) + \text{binomM}(t - 1, b). \]
\[ \text{return } D[t, b] \]

Assuming that every arithmetic operation takes \( O(1) \) time, What is the running time of \( \text{binomM}(n, \lfloor n/2 \rfloor) \)?

- \( \Theta(1) \).
- \( \Theta(n) \).
- \( \Theta(n^2) \).
- \( \Theta(n^3) \).
- \( \Theta\left(\binom{n}{\lfloor n/2 \rfloor}\right) \).
THE END

...  

(for now)
13.2

Dynamic programming
Removing the recursion by filling the table in the right order

“Dynamic programming”

\[
\text{Fib}(n) :
\begin{align*}
\text{if} \ (n = 0) & \quad \text{return} \ 0 \\
\text{if} \ (n = 1) & \quad \text{return} \ 1 \\
\text{if} \ (M[n] \neq -1) & \quad \text{return} \ M[n] \\
M[n] & \leftarrow \text{Fib}(n-1) + \text{Fib}(n-2) \\
\text{return} \ M[n]
\end{align*}
\]

\[
\text{FibIter}(n) :
\begin{align*}
\text{if} \ (n = 0) & \quad \text{return} \ 0 \\
\text{if} \ (n = 1) & \quad \text{return} \ 1 \\
F[0] & = 0 \\
F[1] & = 1 \\
\text{for} \ i = 2 \ \text{to} \ n \ \text{do} & \\
& \quad F[i] = F[i - 1] + F[i - 2] \\
\text{return} \ F[n]
\end{align*}
\]
Dynamic programming: Saving space!

Saving space. Do we need an array of \( n \) numbers? Not really.

\[
\text{FibIter}(n): \\
\text{if } (n = 0) \text{ then} \\
\quad \text{return } 0 \\
\text{if } (n = 1) \text{ then} \\
\quad \text{return } 1 \\
F[0] = 0 \\
F[1] = 1 \\
\text{for } i = 2 \text{ to } n \text{ do} \\
\quad F[i] = F[i - 1] + F[i - 2] \\
\text{return } F[n]
\]

\[
\text{FibIter}(n): \\
\text{if } (n = 0) \text{ then} \\
\quad \text{return } 0 \\
\text{if } (n = 1) \text{ then} \\
\quad \text{return } 1 \\
prev2 = 0 \\
prev1 = 1 \\
\text{for } i = 2 \text{ to } n \text{ do} \\
\quad \text{temp } = \text{prev1 } + \text{prev2} \\
\quad prev2 = \text{prev1} \\
\quad prev1 = \text{temp} \\
\text{return } prev1
\]
Dynamic programming – quick review

Dynamic Programming is **smart recursion**
+ explicit memoization
+ filling the table in right order
+ removing recursion.
Dynamic programming – quick review

Dynamic Programming is smart recursion
+ explicit memoization
+ filling the table in right order
+ removing recursion.
Dynamic programming – quick review

Dynamic Programming is smart recursion
+ explicit memoization
+ filling the table in right order
+ removing recursion.
Analyzing memoized recursive function

**Question:** Suppose we have a recursive program \( \text{foo}(x) \) that takes an input \( x \).

- On input of size \( n \) the number of distinct sub-problems that \( \text{foo}(x) \) generates is at most \( A(n) \)
- \( \text{foo}(x) \) spends at most \( B(n) \) time not counting the time for its recursive calls.

Suppose we memoize the recursion.

**Assumption:** Storing and retrieving solutions to pre-computed problems takes \( O(1) \) time.

**Q:** What is an upper bound on the running time of the memoized version of \( \text{foo}(x) \) if \( |x| = n \)?

\( O(A(n)B(n)) \).
Analyzing memoized recursive function

**Question:** Suppose we have a recursive program $\text{foo}(x)$ that takes an input $x$.
- On input of size $n$ the number of distinct sub-problems that $\text{foo}(x)$ generates is at most $A(n)$
- $\text{foo}(x)$ spends at most $B(n)$ time not counting the time for its recursive calls.

Suppose we memoize the recursion.

**Assumption:** Storing and retrieving solutions to pre-computed problems takes $O(1)$ time.

**Q:** What is an upper bound on the running time of memoized version of $\text{foo}(x)$ if $|x| = n$? $O(A(n)B(n))$.  

Analyzing memoized recursive function

**Question:** Suppose we have a recursive program $\text{foo}(x)$ that takes an input $x$.
- On input of size $n$ the number of distinct sub-problems that $\text{foo}(x)$ generates is at most $A(n)$
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Analyzing memoized recursive function

**Question:** Suppose we have a recursive program \( \text{foo}(x) \) that takes an input \( x \).

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Suppose we memoize the recursion.

**Assumption:** Storing and retrieving solutions to pre-computed problems takes \( O(1) \) time.

**Q:** What is an upper bound on the running time of memoized version of \( \text{foo}(x) \) if \( |x| = n \)? \( O(A(n)B(n)) \).
13.2.1
Fibonacci numbers are big – corrected running time analysis
Back to Fibonacci Numbers

Is the iterative algorithm a polynomial time algorithm? Does it take \( O(n) \) time?

1. input is \( n \) and hence input size is \( \Theta(\log n) \)
2. output is \( F(n) \) and output size is \( \Theta(n) \). Why?
3. Hence output size is exponential in input size so no polynomial time algorithm possible!
4. Running time of iterative algorithm: \( \Theta(n) \) additions but number sizes are \( O(n) \) bits long! Hence total time is \( O(n^2) \), in fact \( \Theta(n^2) \). Why?

**FibIter**\( (n) \):
- if \( (n = 0) \) then
  - return 0
- if \( (n = 1) \) then
  - return 1
  prev\_2 = 0
  prev\_1 = 1
  for \( i = 2 \) to \( n \) do
    temp = prev\_1 + prev\_2
    prev\_2 = prev\_1
    prev\_1 = temp
  return prev\_1
THE END

...

(for now)
13.3
Checking if a string is in $L^*$
Problem

Input A string \( w \in \Sigma^* \) and access to a language \( L \subseteq \Sigma^* \) via function \( \text{IsInL}(\text{string } x) \) that decides whether \( x \) is in \( L \)

Goal Decide if \( w \in L^* \) using \( \text{IsInL}(\text{string } x) \) as a black box sub-routine
Problem

Input  A string $w \in \Sigma^*$ and access to a language $L \subseteq \Sigma^*$ via function $\text{IsInL}(\text{string } x)$ that decides whether $x$ is in $L$

Goal  Decide if $w \in L$ using $\text{IsInL}(\text{string } x)$ as a black box sub-routine
Problem

Input A string $w \in \Sigma^*$ and access to a language $L \subseteq \Sigma^*$ via function $\text{IsInL}(\text{string } x)$ that decides whether $x$ is in $L$.

Goal Decide if $w \in L$ using $\text{IsInL}(\text{string } x)$ as a black box sub-routine.
Problem

Input  A string $w \in \Sigma^*$ and access to a language $L \subseteq \Sigma^*$ via function $\text{IsInL}(\text{string } x)$ that decides whether $x$ is in $L$

Goal  Decide if $w \in L$ using $\text{IsInL}(\text{string } x)$ as a black box sub-routine
Problem

Input A string $w \in \Sigma^*$ and access to a language $L \subseteq \Sigma^*$ via function $\text{IsInL}(\text{string } x)$ that decides whether $x$ is in $L$.
Problem

Input  A string $w \in \Sigma^*$ and access to a language $L \subseteq \Sigma^*$ via function $\text{IsInL}(\text{string } x)$ that decides whether $x$ is in $L$

Goal  Decide if $w \in L^*$ using $\text{IsInL}(\text{string } x)$ as a black box sub-routine
Problem

Input  A string $w \in \Sigma^*$ and access to a language $L \subseteq \Sigma^*$ via function $\text{IsInL}(\text{string } x)$ that decides whether $x$ is in $L$

Goal  Decide if using $\text{IsInL}(\text{string } x)$ as a black box sub-routine

Example 13.1.
Suppose $L$ is $\text{English}$ and we have a procedure to check whether a string/word is in the $\text{English}$ dictionary.

- Is the string “isthisanenglishsentence” in $\text{English}$*? 
- Is “stampstamp” in $\text{English}$*? 
- Is “zibzzzad” in $\text{English}$*?
Recursive Solution

When is \( w \in L^* \)?

\[ w \in L^* \iff w \in L \text{ or if } w = uv \text{ where } u \in L^* \text{ and } v \in L, \ |v| \geq 1. \]

Assume \( w \) is stored in array \( A[1..n] \)

\[
\text{IsInL}^*(A[1..n]):
\]
\[
\begin{align*}
\text{If } (n = 0) \text{ Output YES} \\
\text{If } (\text{IsInL}(A[1..n])) \\
\quad \text{Output YES} \\
\text{Else} \\
\quad \text{For } (i = 1 \text{ to } n - 1) \text{ do} \\
\quad \quad \text{If } \text{IsInL}^*(A[1..i]) \text{ and } \text{IsInL}(A[i + 1..n]) \\
\quad \quad \quad \text{Output YES} \\
\end{align*}
\]

Output NO
Recursive Solution

When is $w \in L^*$?

$w \in L^* \iff w \in L$ or if $w = uv$ where $u \in L^*$ and $v \in L$, $|v| \geq 1$.

Assume $w$ is stored in array $A[1..n]$

Let $\text{IsInL}^*(A[1..n])$:
- If ($n = 0$) Output YES
- If ($\text{IsInL}(A[1..n])$)
  - Output YES
- Else
  - For ($i = 1$ to $n - 1$) do
    - If $\text{IsInL}^*(A[1..i])$ and $\text{IsInL}(A[i + 1..n])$
      - Output YES
  - Output NO
Recursive Solution

When is $w \in L^*$?

\[ w \in L^* \iff w \in L \text{ or if } w = uv \text{ where } u \in L^* \text{ and } v \in L, \ |v| \geq 1. \]

Assume $w$ is stored in array $A[1..n]$

```
IsInL*(A[1..n]):
    If (n = 0) Output YES
    If (IsInL(A[1..n]))
        Output YES
    Else
        For (i = 1 to n - 1) do
            If IsInL*(A[1..i]) and IsInL(A[i + 1..n])
                Output YES
        Output NO
```
Recursive Solution

Assume \( w \) is stored in array \( A[1..n] \)

\[
\text{IsInL}^*(A[1..n]):
\begin{align*}
\text{If } (n = 0) & \text{ Output YES} \\
\text{If } (\text{IsInL}(A[1..n])) & \text{ Output YES} \\
\text{Else} & \\
\text{For } (i = 1 \text{ to } n - 1) & \text{ do} \\
& \text{If } \text{IsInL}^*(A[1..i]) \text{ and IsInL}(A[i+1..n]) \\
& \text{ Output YES} \\
\text{Output NO}
\end{align*}
\]

Question: How many distinct sub-problems does \( \text{IsInL}^*(A[1..n]) \) generate? \( O(n) \)
Recursive Solution

Assume \( w \) is stored in array \( A[1..n] \)

\[
\text{IsInL}^*(A[1..n]): \\
\text{If } (n = 0) \text{ Output YES} \\
\text{If } (\text{IsInL}(A[1..n])) \text{ Output YES} \\
\text{Else} \\
\text{For } (i = 1 \text{ to } n - 1) \text{ do} \\
\quad \text{If } \text{IsInL}^*(A[1..i]) \text{ and } \text{IsInL}(A[i + 1..n]) \text{ Output YES} \\
\text{Output NO}
\]

**Question:** How many distinct sub-problems does \( \text{IsInL}^*(A[1..n]) \) generate? \( O(n) \)
Recursive Solution

Assume \( w \) is stored in array \( A[1..n] \)

\[
\text{IsInL}^*(A[1..n]):
\]
- If \((n = 0)\) Output YES
- If \((\text{IsInL}(A[1..n]))\)
  - Output YES
- Else
  - For \((i = 1 \text{ to } n - 1)\) do
    - If \(\text{IsInL}^*(A[1..i]) \text{ and IsInL}(A[i + 1..n])\)
      - Output YES
- Output NO

**Question:** How many distinct sub-problems does \(\text{IsInL}^*(A[1..n])\) generate? \(O(n)\)
Example

Consider string \textit{samiam}
Naming subproblems and recursive equation

After seeing that number of subproblems is $O(n)$ we name them to help us understand the structure better.

$\text{ISL}^*(i)$: a boolean which is 1 if $A[1..i]$ is in $L^*$, 0 otherwise

Base case: $\text{ISL}^*(0) = 1$ interpreting $A[1..0]$ as $\epsilon$

Recursive relation:

- $\text{ISL}^*(i) = 1$ if $\exists j, 0 \leq j < i$ s.t. $\text{ISL}^*(j)$ and $\text{IsInL}(A[j + 1..i])$
- $\text{ISL}^*(i) = 0$ otherwise

Output: $\text{ISL}^*(n)$
Naming subproblems and recursive equation

After seeing that number of subproblems is $O(n)$ we name them to help us understand the structure better.

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Recursive relation:

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- $\text{ISL}^*(i) = 0$ otherwise

Output: $\text{ISL}^*(n)$
Naming subproblems and recursive equation

After seeing that number of subproblems is $O(n)$ we name them to help us understand the structure better.

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Recursive relation:

- $\text{ISL}^*(i) = 1$ if $\exists j, 0 \leq j < i$ s.t $\text{ISL}^*(j)$ and $\text{IsInL}(A[j + 1..i])$
- $\text{ISL}^*(i) = 0$ otherwise

Output: $\text{ISL}^*(n)$
Removing recursion to obtain iterative algorithm

Typically, after finding a dynamic programming recursion, we often convert the recursive algorithm into an iterative algorithm via explicit memoization and bottom up computation.

Why? Mainly for further optimization of running time and space.

How?
- First, allocate a data structure (usually an array or a multi-dimensional array that can hold values for each of the subproblems)
- Figure out a way to order the computation of the sub-problems starting from the base case.

Caveat: Dynamic programming is not about filling tables. It is about finding a smart recursion. First, find the correct recursion.
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Removing recursion to obtain iterative algorithm

Typically, after finding a dynamic programming recursion, we often convert the recursive algorithm into an **iterative** algorithm via **explicit memoization** and **bottom up** computation.

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- First, allocate a data structure (usually an array or a multi-dimensional array that can hold values for each of the subproblems)
- Figure out a way to order the computation of the sub-problems starting from the base case.

Caveat: Dynamic programming is not about filling tables. It is about finding a smart recursion. First, find the correct recursion.
Iterative Algorithm

\[\text{Iterative Algorithm}\]

\[
\text{IsStringinLstar-Iterative}(A[1..n]):
\]

\[
\begin{align*}
\text{boolean ISL}[0..(n + 1)] & \\
\text{ISL}[0] = \text{TRUE} & \\
\text{for } i = 1 \text{ to } n \text{ do} & \\
\quad & \\
\quad \text{for } j = 0 \text{ to } i - 1 \text{ do} & \\
\quad \quad & \\
\quad \quad \text{if (ISL}[j] \text{ and IsInL}(A[j + 1..i])) & \\
\quad \quad \quad \text{ISL}[i] = \text{TRUE} & \\
\quad \quad \quad \text{break} & \\
\text{end for} & \\
\text{end for} & \\
\text{if (ISL}[n] = 1) \text{ Output YES} & \\
\text{else} \text{ Output NO} & \\
\end{align*}
\]

- **Running time:** \(O(n^2)\) (assuming call to IsInL is \(O(1)\) time)
- **Space:** \(O(n)\)
Iterative Algorithm

IsStringinLstar-Iterative($A[1..n]$):

boolean $ISL^*[0..(n + 1)]$

$ISL^*[0] = TRUE$

for $i = 1$ to $n$ do

  for $j = 0$ to $i - 1$ do

    if ($ISL^*[j]$ and $IsInL(A[j + 1..i])$)
      $ISL^*[i] = TRUE$
      break

if ($ISL^*[n] = 1$) Output YES
else Output NO

- Running time: $O(n^2)$ (assuming call to $IsInL$ is $O(1)$ time)
- Space: $O(n)$
Iterative Algorithm

**IsStringinLstar-Iterative**($A[1..n]$):

- boolean $ISL^*[0..(n + 1)]$
- $ISL^*[0] = TRUE$
- for $i = 1$ to $n$ do
  - for $j = 0$ to $i - 1$ do
    - if ($ISL^*[j]$ and $IsInL(A[j + 1..i])$)
      - $ISL^*[i] = TRUE$
      - break
- if ($ISL^*[n] = 1$) Output YES
  - else Output NO

- **Running time**: $O(n^2)$ (assuming call to $IsInL$ is $O(1)$ time)
- **Space**: $O(n)$
Iterative Algorithm

**IsStringinLstar-Iterative**($A[1..n]$):

boolean $\text{ISL}^*[0..(n + 1)]$

$\text{ISL}^*[0] = \text{TRUE}$

for $i = 1$ to $n$ do
    for $j = 0$ to $i - 1$ do
        if ($\text{ISL}^*[j]$ and $\text{IsInL}(A[j + 1..i])$)
            $\text{ISL}^*[i] = \text{TRUE}$
            break

if ($\text{ISL}^*[n] = 1$) Output YES
else Output NO

- **Running time:** $O(n^2)$ (assuming call to $\text{IsInL}$ is $O(1)$ time)
- **Space:** $O(n)$
Iterative Algorithm

\[
\text{IsStringinLstar-Iterative}(A[1..n]): \\
\begin{align*}
\text{boolean } & \text{ISL}^*[0..(n+1)] \\
\text{ISL}^*[0] & = \text{TRUE} \\
\text{for } i & = 1 \text{ to } n \text{ do} \\
\text{for } j & = 0 \text{ to } i - 1 \text{ do} \\
& \text{if (ISL}^*[j] \text{ and IsInL}(A[j+1..i])\text{)} \\
& \quad \text{ISL}^*[i] = \text{TRUE} \\
& \quad \text{break} \\
\text{if (ISL}^*[n] = 1\text{) Output YES} \\
\text{else Output NO}
\end{align*}
\]

- **Running time**: \(O(n^2)\) (assuming call to \text{IsInL} is \(O(1)\) time)
- **Space**: \(O(n)\)
Example

Consider string \textit{samiam}
THE END

...

(for now)
13.4
Longest Increasing Subsequence Revisited
13.4.1

Longest Increasing Subsequence
Sequences

Definition 13.1.
**Sequence**: an ordered list $a_1, a_2, \ldots, a_n$. **Length** of a sequence is number of elements in the list.

Definition 13.2.
$a_{i_1}, \ldots, a_{i_k}$ is a **subsequence** of $a_1, \ldots, a_n$ if $1 \leq i_1 < i_2 < \ldots < i_k \leq n$.

Definition 13.3.
A sequence is **increasing** if $a_1 < a_2 < \ldots < a_n$. It is **non-decreasing** if $a_1 \leq a_2 \leq \ldots \leq a_n$. Similarly **decreasing** and **non-increasing**.
Example 13.4.

1. Sequence: 6, 3, 5, 2, 7, 8, 1, 9
2. Subsequence of above sequence: 5, 2, 1
3. Increasing sequence: 3, 5, 9, 17, 54
4. Decreasing sequence: 34, 21, 7, 5, 1
5. Increasing subsequence of the first sequence: 2, 7, 9.
Longest Increasing Subsequence Problem

**Input** A sequence of numbers $a_1, a_2, \ldots, a_n$

**Goal** Find an *increasing subsequence* $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ of maximum length

Example 13.5.

1. Sequence: 6, 3, 5, 2, 7, 8, 1
2. Increasing subsequences: 6, 7, 8 and 3, 5, 7, 8 and 2, 7 etc
3. Longest increasing subsequence: 3, 5, 7, 8
Longest Increasing Subsequence Problem

Input  A sequence of numbers $a_1, a_2, \ldots, a_n$

Goal  Find an increasing subsequence $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ of maximum length

Example 13.5.

1. Sequence: 6, 3, 5, 2, 7, 8, 1
2. Increasing subsequences: 6, 7, 8 and 3, 5, 7, 8 and 2, 7 etc
3. Longest increasing subsequence: 3, 5, 7, 8
Recursive Approach: Take 1

LIS: Longest increasing subsequence

Can we find a recursive algorithm for LIS?

\[ \text{LIS}(A[1..n]):\]

1. Case 1: Does not contain \( A[n] \) in which case
   \[ \text{LIS}(A[1..n]) = \text{LIS}(A[1..(n-1)]) \]

2. Case 2: contains \( A[n] \) in which case \( \text{LIS}(A[1..n]) \) is not so clear.

Observation 13.6.

For second case we want to find a subsequence in \( A[1..(n-1)] \) that is restricted to numbers less than \( A[n] \). This suggests that a more general problem is \( \text{LIS}\_\text{smaller}(A[1..n], x) \) which gives the longest increasing subsequence in \( A \) where each number in the sequence is less than \( x \).
Recursive Approach: Take 1

LIS: Longest increasing subsequence

Can we find a recursive algorithm for LIS?

\[ \text{LIS}(A[1..n]): \]

1. Case 1: Does not contain \( A[n] \) in which case
   \[ \text{LIS}(A[1..n]) = \text{LIS}(A[1..(n-1)]) \]
2. Case 2: contains \( A[n] \) in which case \( \text{LIS}(A[1..n]) \) is not so clear.

Observation 13.6.

For second case we want to find a subsequence in \( A[1..(n-1)] \) that is restricted to numbers less than \( A[n] \). This suggests that a more general problem is
\[ \text{LIS\_smaller}(A[1..n], x) \]
which gives the longest increasing subsequence in \( A \) where each number in the sequence is less than \( x \).
Recursive Approach

\(LIS(A[1..n])\): the length of longest increasing subsequence in \(A\)

\(LIS_{\text{smaller}}(A[1..n], x)\): length of longest increasing subsequence in \(A[1..n]\) with all numbers in subsequence less than \(x\)

\[
\text{LIS}_{\text{smaller}}(A[1..i], x) : \begin{align*}
\text{if } i &= 0 \text{ then return } 0 \\
m &= \text{LIS}_{\text{smaller}}(A[1..i-1], x) \\
\text{if } A[i] &< x \text{ then} \\
&\quad m = \max(m, 1 + \text{LIS}_{\text{smaller}}(A[1..i-1], A[i])) \\
\text{Output } m
\end{align*}
\]

\[
\text{LIS}(A[1..n]) : \quad \text{return } \text{LIS}_{\text{smaller}}(A[1..n], \infty)
\]
Recursive Approach

\[
\text{LIS}_{\text{smaller}}(A[1..i], x): \\
\quad \text{if } i = 0 \text{ then return } 0 \\
\quad m = \text{LIS}_{\text{smaller}}(A[1..i-1], x) \\
\quad \text{if } A[i] < x \text{ then} \\
\quad \quad m = \max(m, 1 + \text{LIS}_{\text{smaller}}(A[1..i-1], A[i])) \\
\quad \text{Output } m
\]

\[
\text{LIS}(A[1..n]): \\
\quad \text{return } \text{LIS}_{\text{smaller}}(A[1..n], \infty)
\]

- How many distinct sub-problems will \(\text{LIS}_{\text{smaller}}(A[1..n], \infty)\) generate? \(O(n^2)\)
- What is the running time if we memoize recursion? \(O(n^2)\) since each call takes \(O(1)\) time to assemble the answers from recursive calls and no other computation.
- How much space for memoization? \(O(n^2)\)
Recursive Approach

\[ \text{LIS}_{\text{smaller}}(A[1..i], x) : \]
\[ \text{if } i = 0 \text{ then return } 0 \]
\[ m = \text{LIS}_{\text{smaller}}(A[1..i - 1], x) \]
\[ \text{if } A[i] < x \text{ then} \]
\[ m = \max(m, 1 + \text{LIS}_{\text{smaller}}(A[1..i - 1], A[i])) \]
\[ \text{Output } m \]

\[ \text{LIS}(A[1..n]) : \]
\[ \text{return } \text{LIS}_{\text{smaller}}(A[1..n], \infty) \]

- How many distinct sub-problems will \( \text{LIS}_{\text{smaller}}(A[1..n], \infty) \) generate? \( O(n^2) \)
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Recursive Approach

\[
\text{LIS\_smaller}(A[1..i], x):
\]
\[
\text{if } i = 0 \text{ then return } 0
\]
\[
m = \text{LIS\_smaller}(A[1..i-1], x)
\]
\[
\text{if } A[i] < x \text{ then }
\]
\[
m = \max(m, 1 + \text{LIS\_smaller}(A[1..i-1], A[i]))
\]
Output \( m \)

\[
\text{LIS}(A[1..n]):
\]
\[
\text{return LIS\_smaller}(A[1..n], \infty)
\]

- How many distinct sub-problems will \( \text{LIS\_smaller}(A[1..n], \infty) \) generate? \( O(n^2) \)
- What is the running time if we memoize recursion? \( O(n^2) \) since each call takes \( O(1) \) time to assemble the answers from to recursive calls and no other computation.
- How much space for memoization? \( O(n^2) \)
Recursive Approach

**LISsmaller**(A[1..i], x):
  if i = 0 then return 0
  m = **LISsmaller**(A[1..i - 1], x)
  if A[i] < x then
    m = max(m, 1 + **LISsmaller**(A[1..i - 1], A[i]))
  Output m

**LIS**(A[1..n]):
  return **LISsmaller**(A[1..n], ∞)

- How many distinct sub-problems will **LISsmaller**(A[1..n], ∞) generate? $O(n^2)$
- What is the running time if we memoize recursion? $O(n^2)$ since each call takes $O(1)$ time to assemble the answers from to recursive calls and no other computation.
- How much space for memoization? $O(n^2)$
Recursive Approach

\[
\text{LIS\_smaller}(A[1..i], x) :
\begin{align*}
&\text{if } i = 0 \text{ then return } 0 \\
&m = \text{LIS\_smaller}(A[1..i - 1], x) \\
&\text{if } A[i] < x \text{ then} \\
&m = \max(m, 1 + \text{LIS\_smaller}(A[1..i - 1], A[i])) \\
\end{align*}
\]

Output \( m \)

\[
\text{LIS}(A[1..n]) :
\begin{align*}
&\text{return LIS\_smaller}(A[1..n], \infty) \\
\end{align*}
\]

- How many distinct sub-problems will \( \text{LIS\_smaller}(A[1..n], \infty) \) generate? \( O(n^2) \)
- What is the running time if we memoize recursion? \( O(n^2) \) since each call takes \( O(1) \) time to assemble the answers from to recursive calls and no other computation.
- How much space for memoization? \( O(n^2) \)
Recursive Approach

\[
\text{LIS\_smaller}(A[1..i], x):
\]
\[
\text{if } i = 0 \text{ then return 0}
\]
\[
m = \text{LIS\_smaller}(A[1..i - 1], x)
\]
\[
\text{if } A[i] < x \text{ then}
\]
\[
m = \max(m, 1 + \text{LIS\_smaller}(A[1..i - 1], A[i]))
\]
\[
\text{Output } m
\]

\[
\text{LIS}(A[1..n]):
\]
\[
\text{return LIS\_smaller}(A[1..n], \infty)
\]

- How many distinct sub-problems will \text{LIS\_smaller}(A[1..n], \infty) generate? \(O(n^2)\)
- What is the running time if we memoize recursion? \(O(n^2)\) since each call takes \(O(1)\) time to assemble the answers from recursive calls and no other computation.
- How much space for memoization? \(O(n^2)\)
Naming subproblems and recursive equation

After seeing that number of subproblems is $O(n^2)$ we name them to help us understand the structure better. For notational ease we add $\infty$ at end of array (in position $n + 1$)

$LIS(i, j)$: length of longest increasing sequence in $A[1..i]$ among numbers less than $A[j]$ (defined only for $i < j$)

Base case: $LIS(0, j) = 0$ for $1 \leq j \leq n + 1$

Recursive relation:

- $LIS(i, j) = LIS(i - 1, j)$ if $A[i] > A[j]$
- $LIS(i, j) = \max\{LIS(i - 1, j), 1 + LIS(i - 1, i)\}$ if $A[i] \leq A[j]$

Output: $LIS(n, n + 1)$. 
Naming subproblems and recursive equation

After seeing that number of subproblems is \( O(n^2) \) we name them to help us understand the structure better. For notational ease we add \( \infty \) at end of array (in position \( n + 1 \))

\[
\text{LIS}(i, j): \text{length of longest increasing sequence in } A[1..i] \text{ among numbers less than } A[j] \text{ (defined only for } i < j) \]

**Base case:** \( \text{LIS}(0, j) = 0 \) for \( 1 \leq j \leq n + 1 \)

**Recursive relation:**
- \( \text{LIS}(i, j) = \text{LIS}(i - 1, j) \) if \( A[i] > A[j] \)
- \( \text{LIS}(i, j) = \max\{\text{LIS}(i - 1, j), 1 + \text{LIS}(i - 1, i)\} \) if \( A[i] \leq A[j] \)

**Output:** \( \text{LIS}(n, n + 1). \)
How to order bottom up computation?

Recursive relation:

\[
LIS(i, j) =
\begin{cases}
0 & \text{if } i = 0 \\
LIS(i - 1, j) & \text{if } A[i] > A[j] \\
\max \left\{ LIS(i - 1, j), 1 + LIS(i - 1, i) \right\} & \text{otherwise}
\end{cases}
\]

Sequence: \( A[1..7] = 6, 3, 5, 2, 7, 8, 1 \)
Iterative algorithm
The dynamic program for longest increasing subsequence

**LIS-Iterative**($A[1..n]$):

1. $A[n + 1] = \infty$
2. int $LIS[0..n, 1..n + 1]$
3. for $j = 1 \ldots n + 1$ do $LIS[0, j] = 0$

4. for $i = 1 \ldots n$ do
5.   for ($j = i + 1 \ldots n$ do
7.       $LIS[i, j] = LIS[i - 1, j]$
8.     else
9.       $LIS[i, j] = \max(LIS[i - 1, j], 1 + LIS[i - 1, i])$

Return $LIS[n, n + 1]$

**Running time:** $O(n^2)$
**Space:** $O(n^2)$
Two comments

**Question:** Can we compute an optimum solution and not just its value?
Yes! See notes.

**Question:** Is there a faster algorithm for LIS? Yes! Using a different recursion and optimizing one can obtain an $O(n \log n)$ time and $O(n)$ space algorithm. $O(n \log n)$ time is not obvious. Depends on improving time by using data structures on top of dynamic programming.
Two comments

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THE END

... (for now)
13.5

How to come up with dynamic programming algorithm: summary
Dynamic Programming

1. Find a “smart” recursion for the problem in which the number of distinct subproblems is small; polynomial in the original problem size.

2. Estimate the number of subproblems, the time to evaluate each subproblem and the space needed to store the value.

3. This gives an upper bound on the total running time if we use automatic/explicit memoization.


5. Eliminate recursion and find an iterative algorithm.

6. ...need to find the right way or order the subproblems evaluation. This leads to a dynamic programming algorithm.

7. Optimize the resulting algorithm further

8. ...

9. Get rich!
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Come up with an explicit memoization algorithm for the problem.

Eliminate recursion and find an iterative algorithm.

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...  

(for now)
13.6 Supplemental: Some experiments with memoization
Edit distance: different memoizations

<table>
<thead>
<tr>
<th>Input size n</th>
<th>Running time in seconds</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>DP</td>
</tr>
<tr>
<td>1,250</td>
<td>0.01</td>
</tr>
<tr>
<td>2,500</td>
<td>0.04</td>
</tr>
<tr>
<td>5,000</td>
<td>0.18</td>
</tr>
<tr>
<td>10,000</td>
<td>0.72</td>
</tr>
<tr>
<td>20,000</td>
<td>2.88</td>
</tr>
<tr>
<td>40,000</td>
<td>12.00</td>
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</tbody>
</table>

For the input $n$, two random strings of length $n$ were generated, and their distance computed using edit distance. Note, that edit-distance is simple enough to that DP gets very good performance. For more complicated problems, the advantage of DP would probably be much smaller. The asymptotic running time here is $\Theta(n^2)$. 
Edit distance: different memoizations

More details

1. The implementation was done in C++, using -O9 in compilation.
2. DP = Dynamic Programming = iterative implementation using arrays.
3. Partial memoization = Still uses recursive code, but remembers the results in tables that are managed directly by the code.
4. Implicit memoization = implemented using the standard unordered_map.
Conclusions

1. If you are in interview setup, you should probably solve the problem using DP. That’s what you would be expected to do.

2. Otherwise, I would probably implement partial memoization – it still has the simplicity of the recursive solution, while having a decent performance. If I really care about performance I would implement the DP.

3. Using implicit memoization probably makes sense only if running time is not really an issue.
THE END

...

(for now)
13.7
Tangential: Fibonacci and his numbers
Fibonacci = Leonardo Bonacci

1. CE 1170–1250.
2. Italian. Spent time in Bugia, Algeria with his father (trader).
3. Traveled around the Mediterranean coast, learned the Hindu–Arabic numerals
4. Hindu–Arabic numerals:
   1. Developed before 400 CE by Hindu philosophers.
   2. Arrived to the Arab world sometime before 825CE.
   3. Muhammad ibn Musa al-Khwarizmi (Algorithm/Algebra) wrote a book in 825 CE explaining the new system. (Showed how to solved quadratic equations.)
5. 1202 CE: Fibonacci wrote a book “Liber Abaci” (book of calculations) that popularized the new system.
6. Brought and popularized the Hindu–Arabic system to Italy.
Fibonacci numbers

Fibonacci in Liber Abaci posed and solved a problem involving the growth of a population of rabbits based on idealized assumptions.

Describe growth processes.

Every month a mature pair of rabbits give birth to one pair of young rabbits.

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Fibonacci numbers II

1. \( \lim_{n \to \infty} \frac{F_n}{F_{n-1}} = \varphi. \)

2. Golden ratio: \( \varphi = \frac{\sqrt{5} + 1}{2} \approx 1.618033. \)

3. For \( a > b > 0, \) \( \varphi = \frac{a+b}{a} = \frac{a}{b}. \) \( \implies \frac{\varphi + 1}{\varphi} = \varphi. \) \( \implies 0 = \varphi^2 - \varphi - 1. \)

4. \( \varphi = \frac{1 \pm \sqrt{1+4}}{2} \) since \( \varphi \) is not negative, so...

5. \( F_n = \frac{\varphi^n - (1-\varphi)^n}{\sqrt{5}} \)

6. Golden ratio goes back to Euclid

7. Many applications of GR and Fibonacci numbers in nature, models (stock market), art, etc...
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\[ \begin{array}{c}
\text{a} \\
\text{b} \\
\text{a+b} \\
\text{a+b is to a as a is to b}
\end{array} \]

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\[
\begin{array}{ccc}
a & b & a+b \\
\hline
\frac{a}{b} & \frac{a+b}{a} & \frac{a+b}{b}
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\( a+b \) is to \( a \) as \( a \) is to \( b \)

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Fibonacci numbers: Binet’s formula

1. \( \varphi = \frac{1+\sqrt{5}}{2} \) and \( \psi = \frac{1-\sqrt{5}}{2} = 1 - \varphi \) are solution to the equation:
\( x^2 = x + 1. \)

2. As such, \( \varphi \) and \( \psi \) a solution to the equation: \( x^n = x^{n-1} + x^{n-2}. \)

3. Consider the sequence \( U_n = U_{n-1} + U_{n-2}. \)
For any \( \alpha, \beta \in \mathbb{R}, \) consider \( U_n = \alpha \varphi^n + \beta \psi^n. \) A valid solution to the sequence.

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   For any \( \alpha, \beta \in \mathbb{R} \), consider \( U_n = \alpha \varphi^n + \beta \psi^n \). A valid solution to the sequence.

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U_n = U_{n-1} + U_{n-2} = \alpha \varphi^{n-1} + \beta \psi^{n-1} + \alpha \varphi^{n-2} + \beta \psi^{n-2}
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Fibonacci numbers: Binet’s formula

1. \( \varphi = \frac{1+\sqrt{5}}{2} \) and \( \psi = \frac{1-\sqrt{5}}{2} = 1 - \varphi \) are solution to the equation: \( x^2 = x + 1. \)

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4. Solve the system
   \( U_0 = 0 \) and \( U_1 = 1 \) \( \iff \) \( \alpha \varphi^0 + \beta \psi^0 = 0 \) and \( \alpha \varphi^1 + \beta \psi^1 = 1 \).
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1. \( \varphi = \frac{1 + \sqrt{5}}{2} \) and \( \psi = \frac{1 - \sqrt{5}}{2} = 1 - \varphi \) are solution to the equation: 
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   \[ U_0 = 0 \text{ and } U_1 = 1 \iff \alpha \varphi^0 + \beta \psi^0 = 0 \text{ and } \alpha \varphi^1 + \beta \psi^1 = 1 \iff \beta = -\alpha \]
Fibonacci numbers: Binet’s formula

1. $\varphi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2} = 1 - \varphi$ are solution to the equation: $x^2 = x + 1$.

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   $\begin{align*}
   U_n &= U_{n-1} + U_{n-2} = \alpha \varphi^{n-1} + \beta \psi^{n-1} + \alpha \varphi^{n-2} + \beta \psi^{n-2} \\
   &= (\alpha \varphi^{n-1} + \alpha \varphi^{n-2}) + (\beta \psi^{n-1} + \beta \psi^{n-2}) = U_{n-1} + U_{n-2}.
   \end{align*}$

4. Solve the system
   $U_0 = 0$ and $U_1 = 1 \iff \alpha \varphi^0 + \beta \psi^0 = 0$ and $\alpha \varphi^1 + \beta \psi^1 = 1 \implies \\ 
   \beta = -\alpha \iff \varphi - \psi = 1/\alpha$
Fibonacci numbers: Binet’s formula

1. \( \varphi = \frac{1+\sqrt{5}}{2} \) and \( \psi = \frac{1-\sqrt{5}}{2} = 1 - \varphi \) are solution to the equation: \( x^2 = x + 1 \).

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   \( \beta = -\alpha \iff \varphi - \psi = 1/\alpha \iff \frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2} = 1/\alpha \)
Fibonacci numbers: Binet’s formula

1. \( \varphi = \frac{1+\sqrt{5}}{2} \) and \( \psi = \frac{1-\sqrt{5}}{2} \) are solution to the equation: \( x^2 = x + 1 \).

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   \( \beta = -\alpha \) \( \implies \) \( \varphi - \psi = \frac{1}{\alpha} \) \( \implies \) \( \frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2} = 1/\alpha \)
   \( \implies \) \( \alpha = 1/\sqrt{5} \)
Fibonacci numbers: Binet’s formula

1. \( \varphi = \frac{1+\sqrt{5}}{2} \) and \( \psi = \frac{1-\sqrt{5}}{2} = 1 - \varphi \) are solution to the equation: \( x^2 = x + 1 \).

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   \( U_0 = 0 \) and \( U_1 = 1 \) \( \iff \) \( \alpha \varphi^0 + \beta \psi^0 = 0 \) and \( \alpha \varphi^1 + \beta \psi^1 = 1 \) \( \implies \)
   \( \beta = -\alpha \implies \varphi - \psi = 1/\alpha \implies \frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2} = 1/\alpha \)
   \( \implies \alpha = 1/\sqrt{5} \implies F_n = U_n = \alpha \varphi^n + \beta \psi^n = \frac{\varphi^n - (1-\varphi)^n}{\sqrt{5}} \)
Fibonacci numbers really fast

\[
\begin{pmatrix}
 y \\
 x + y
\end{pmatrix}
= 
\begin{pmatrix}
 0 & 1 \\
 1 & 1
\end{pmatrix}
\begin{pmatrix}
 x \\
 y
\end{pmatrix}.
\]

As such,

\[
\begin{pmatrix}
 F_{n-1} \\
 F_n
\end{pmatrix}
= 
\begin{pmatrix}
 0 & 1 \\
 1 & 1
\end{pmatrix}
\begin{pmatrix}
 F_{n-2} \\
 F_{n-1}
\end{pmatrix}
= 
\begin{pmatrix}
 0 & 1 \\
 1 & 1
\end{pmatrix}^2
\begin{pmatrix}
 F_{n-3} \\
 F_{n-2}
\end{pmatrix}
= 
\begin{pmatrix}
 0 & 1 \\
 1 & 1
\end{pmatrix}^{n-3}
\begin{pmatrix}
 F_2 \\
 F_1
\end{pmatrix}.
\]
More on fast Fibonacci numbers

Continued

Thus, computing the \( n \)th Fibonacci number can be done by computing \( \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{n-3} \).

Which can be done in \( O(\log n) \) time (how?). What is wrong?