Proving Non-regularity

Lecture 6
Thursday, September 10, 2020
6.1

Not all languages are regular
Theorem

Languages accepted by **DFA**s, **NFA**s, and regular expressions are the same.

**Question:** Is every language a regular language? No.

- Each DFA $M$ can be represented as a string over a finite alphabet $\Sigma$ by appropriate encoding.
- Hence number of regular languages is **countably infinite**.
- Number of languages is **uncountably infinite**.
- Hence there must be a non-regular language!
### Theorem

Languages accepted by **DFA**s, **NFA**s, and regular expressions are the same.

### Question: Is every language a regular language? No.

- Each DFA $M$ can be represented as a string over a finite alphabet $\Sigma$ by appropriate encoding.
- Hence number of regular languages is **countably infinite**.
- Number of languages is **uncountably infinite**.
- Hence there must be a non-regular language!
Regular Languages, DFAs, NFAs

**Theorem**

*Languages accepted by DFA\(^s\), NFA\(^s\), and regular expressions are the same.*

**Question:** Is every language a regular language? No.

- Each DFA \( M \) can be represented as a string over a finite alphabet \( \Sigma \) by appropriate encoding.
- Hence number of regular languages is countably infinite.
- Number of languages is uncountably infinite.
- Hence there must be a non-regular language!
Theorem

Languages accepted by DFAs, NFAs, and regular expressions are the same.

Question: Is every language a regular language? No.

- Each DFA $M$ can be represented as a string over a finite alphabet $\Sigma$ by appropriate encoding.
- Hence number of regular languages is countably infinite.
- Number of languages is uncountably infinite.
- Hence there must be a non-regular language!
Regular Languages, DFAs, NFAs

Theorem
Languages accepted by DFAs, NFAs, and regular expressions are the same.

Question: Is every language a regular language? No.
- Each DFA $M$ can be represented as a string over a finite alphabet $\Sigma$ by appropriate encoding.
- Hence number of regular languages is countably infinite.
- Number of languages is uncountably infinite.
- Hence there must be a non-regular language!
Regular Languages, DFAs, NFAs

Theorem

Languages accepted by DFAs, NFAs, and regular expressions are the same.

Question: Is every language a regular language? No.

- Each DFA $M$ can be represented as a string over a finite alphabet $\Sigma$ by appropriate encoding
- Hence number of regular languages is countably infinite
- Number of languages is uncountably infinite
- Hence there must be a non-regular language!
A direct proof

\[ L = \{0^i1^i \mid i \geq 0\} = \{\epsilon, 01, 0011, 000111, \ldots\} \]

**Theorem**

\( L \) is not regular.
A Simple and Canonical Non-regular Language

$L = \{0^i1^i \mid i \geq 0\} = \{\epsilon, 01, 0011, 000111, \cdots, \}$

**Theorem**

$L$ is not regular.

**Question:** Proof?

**Intuition:** Any program to recognize $L$ seems to require counting number of zeros in input which cannot be done with fixed memory.

How do we formalize intuition and come up with a formal proof?
A Simple and Canonical Non-regular Language

\[ L = \{0^k1^k \mid i \geq 0\} = \{\varepsilon, 01, 0011, 000111, \ldots\} \]

**Theorem**

\( L \) is not regular.

**Question:** Proof?

**Intuition:** Any program to recognize \( L \) seems to require counting number of zeros in input which cannot be done with fixed memory.

How do we formalize intuition and come up with a formal proof?
A Simple and Canonical Non-regular Language

$L = \{0^i 1^i \mid i \geq 0\} = \{\epsilon, 01, 0011, 000111, \cdots\}$

**Theorem**

$L$ is not regular.

**Question:** Proof?

**Intuition:** Any program to recognize $L$ seems to require counting number of zeros in input which cannot be done with fixed memory.

How do we formalize intuition and come up with a formal proof?
A Simple and Canonical Non-regular Language

$L = \{0^k1^k \mid i \geq 0\} = \{\epsilon, 01, 0011, 000111, \cdots, \}$

**Theorem**

$L$ is not regular.

**Question:** Proof?

**Intuition:** Any program to recognize $L$ seems to require counting number of zeros in input which cannot be done with fixed memory.

How do we formalize intuition and come up with a formal proof?
A Simple and Canonical Non-regular Language

\[ L = \{0^k1^k \mid i \geq 0\} = \{\epsilon, 01, 0011, 000111, \cdots\} \]

**Theorem**

\( L \) is not regular.

**Question:** Proof?

**Intuition:** Any program to recognize \( L \) seems to require counting number of zeros in input which cannot be done with fixed memory.

How do we formalize intuition and come up with a formal proof?
Proof by Contradiction

- Suppose $L$ is regular. Then there is a DFA $M$ such that $L(M) = L$.
- Let $M = (Q, \{0, 1\}, \delta, s, A)$ where $|Q| = n$.

Consider strings $\epsilon, 0, 00, 000, \ldots, 0^n$ total of $n + 1$ strings.

What states does $M$ reach on the above strings? Let $q_i = \delta^*(s, 0^i)$.

By pigeon hole principle $q_i = q_j$ for some $0 \leq i < j \leq n$.
That is, $M$ is in the same state after reading $0^i$ and $0^j$ where $i \neq j$.

$M$ should accept $0^i1^i$ but then it will also accept $0^j1^i$ where $i \neq j$.
This contradicts the fact that $M$ accepts $L$. Thus, there is no DFA for $L$. 
Proof by Contradiction

- Suppose $L$ is regular. Then there is a DFA $M$ such that $L(M) = L$.
- Let $M = (Q, \{0, 1\}, \delta, s, A)$ where $|Q| = n$.

Consider strings $\epsilon, 0, 00, 000, \cdots, 0^n$ total of $n + 1$ strings.

What states does $M$ reach on the above strings? Let $q_i = \delta^*(s, 0^i)$.

By pigeon hole principle $q_i = q_j$ for some $0 \leq i < j \leq n$.
That is, $M$ is in the same state after reading $0^i$ and $0^j$ where $i \neq j$.

$M$ should accept $0^i1^i$ but then it will also accept $0^j1^i$ where $i \neq j$.
This contradicts the fact that $M$ accepts $L$. Thus, there is no DFA for $L$. 
Proof by Contradiction

- Suppose $L$ is regular. Then there is a DFA $M$ such that $L(M) = L$.
- Let $M = (Q, \{0, 1\}, \delta, s, A)$ where $|Q| = n$.

Consider strings $\epsilon, 0, 00, 000, \cdots, 0^n$ total of $n + 1$ strings.

What states does $M$ reach on the above strings? Let $q_i = \delta^*(s, 0^i)$.

By pigeon hole principle $q_i = q_j$ for some $0 \leq i < j \leq n$.
That is, $M$ is in the same state after reading $0^i$ and $0^j$ where $i \neq j$.

$M$ should accept $0^i1^i$ but then it will also accept $0^j1^i$ where $i \neq j$.
This contradicts the fact that $M$ accepts $L$. Thus, there is no DFA for $L$. 
Proof by Contradiction

- Suppose $L$ is regular. Then there is a DFA $M$ such that $L(M) = L$.
- Let $M = (Q, \{0, 1\}, \delta, s, A)$ where $|Q| = n$.

Consider strings $\epsilon, 0, 00, 000, \cdots, 0^n$ total of $n + 1$ strings.

What states does $M$ reach on the above strings? Let $q_i = \delta^*(s, 0^i)$.

By pigeon hole principle $q_i = q_j$ for some $0 \leq i < j \leq n$.
That is, $M$ is in the same state after reading $0^i$ and $0^j$ where $i \neq j$.

$M$ should accept $0^i1^i$ but then it will also accept $0^j1^i$ where $i \neq j$.
This contradicts the fact that $M$ accepts $L$. Thus, there is no DFA for $L$. 
Proof by Contradiction

- Suppose $L$ is regular. Then there is a DFA $M$ such that $L(M) = L$.
- Let $M = (Q, \{0, 1\}, \delta, s, A)$ where $|Q| = n$.

Consider strings $\epsilon, 0, 00, 000, \ldots, 0^n$ total of $n + 1$ strings.

What states does $M$ reach on the above strings? Let $q_i = \delta^*(s, 0^i)$.

By pigeon hole principle $q_i = q_j$ for some $0 \leq i < j \leq n$.
That is, $M$ is in the same state after reading $0^i$ and $0^j$ where $i \neq j$.

$M$ should accept $0^i1^i$ but then it will also accept $0^j1^i$ where $i \neq j$.
This contradicts the fact that $M$ accepts $L$. Thus, there is no DFA for $L$. 

THE END

... (for now)
6.2
When two states are equivalent?
Equivalence between states

Definition

\( M = (Q, \Sigma, \delta, s, A) \): DFA.

Two states \( p, q \in Q \) are equivalent if for all strings \( w \in \Sigma^* \), we have that

\[
\delta^*(p, w) \in A \iff \delta^*(q, w) \in A.
\]

One can merge any two states that are equivalent into a single state.
Distinguishing between states

**Definition**

\[ M = (Q, \Sigma, \delta, s, A): \text{DFA.} \]

Two states \( p, q \in Q \) are distinguishable if there exists a string \( w \in \Sigma^* \), such that

\[ \delta^*(p, w) \in A \quad \text{and} \quad \delta^*(q, w) \notin A. \]

or

\[ \delta^*(p, w) \notin A \quad \text{and} \quad \delta^*(q, w) \in A. \]
Distinguishable prefixes

\( M = (Q, \Sigma, \delta, s, A) \): DFA

**Idea:** Every string \( w \in \Sigma^* \) defines a state \( \nabla w = \delta^*(s, w) \).

**Definition**

Two strings \( u, w \in \Sigma^* \) are distinguishable for \( M \) (or \( L(M) \)) if \( \nabla u \) and \( \nabla w \) are distinguishable.

**Definition (Direct restatement)**

Two prefixes \( u, w \in \Sigma^* \) are distinguishable for a language \( L \) if there exists a string \( x \), such that \( ux \in L \) and \( wx \notin L \) (or \( ux \notin L \) and \( wx \in L \)).
Distinguishable prefixes

\( M = (Q, \Sigma, \delta, s, A) \): DFA

**Idea:** Every string \( w \in \Sigma^* \) defines a state \( \nabla w = \delta^*(s, w) \).

**Definition**

Two strings \( u, w \in \Sigma^* \) are **distinguishable** for \( M \) (or \( L(M) \)) if \( \nabla u \) and \( \nabla w \) are distinguishable.

**Definition (Direct restatement)**

Two prefixes \( u, w \in \Sigma^* \) are distinguishable for a language \( L \) if there exists a string \( x \), such that \( ux \in L \) and \( wx \notin L \) (or \( ux \notin L \) and \( wx \in L \)).
**Distinguishable prefixes**

\( M = (Q, \Sigma, \delta, s, A) \): DFA

**Idea:** Every string \( w \in \Sigma^* \) defines a state \( \nabla w = \delta^*(s, w) \).

**Definition**

Two strings \( u, w \in \Sigma^* \) are distinguishable for \( M \) (or \( L(M) \)) if \( \nabla u \) and \( \nabla w \) are distinguishable.

**Definition (Direct restatement)**

Two prefixes \( u, w \in \Sigma^* \) are distinguishable for a language \( L \) if there exists a string \( x \), such that \( ux \in L \) and \( wx \notin L \) (or \( ux \notin L \) and \( wx \in L \)).
Distinguishable prefixes

\( M = (Q, \Sigma, \delta, s, A) \): DFA

Idea: Every string \( w \in \Sigma^* \) defines a state \( \nabla w = \delta^*(s, w) \).

Definition

Two strings \( u, w \in \Sigma^* \) are distinguishable for \( M \) (or \( L(M) \)) if \( \nabla u \) and \( \nabla w \) are distinguishable.

Definition (Direct restatement)

Two prefixes \( u, w \in \Sigma^* \) are distinguishable for a language \( L \) if there exists a string \( x \), such that \( ux \in L \) and \( wx \notin L \) (or \( ux \notin L \) and \( wx \in L \)).
Distinguishable means different states

Lemma

$L$: regular language.
$M = (Q, \Sigma, \delta, s, A)$: DFA for $L$.

If $x, y \in \Sigma^*$ are distinguishable, then $\nabla x \neq \nabla y$.

Reminder: $\nabla x = \delta^*(s, x) \in Q$ and $\nabla y = \delta^*(s, y) \in Q$
Proof by a figure

Possible

\[ \delta^*(s,x) \]
\[ \delta^*(s,y) \]
\[ \delta^*(s,xw) \]
\[ \delta^*(s,yw) \]

Not possible

\[ \delta^*(s,x) = \delta^*(s,y) \]

Har-Peled (UIUC)
Distinguishable strings means different states: Proof

**Lemma**

\( L \): regular language.

\( M = (Q, \Sigma, \delta, s, A) \): DFA for \( L \).

If \( x, y \in \Sigma^\ast \) are distinguishable, then \( \nabla x \neq \nabla y \).

**Proof.**

Assume for the sake of contradiction that \( \nabla x = \nabla y \).

By assumption \( \exists w \in \Sigma^\ast \) such that \( \nabla xw \in A \) and \( \nabla yw \notin A \).

\[ \implies A \ni \nabla xw = \delta^*(s, xw) = \delta^*(\nabla x, w) = \delta^*(\nabla y, w) = \delta^*(s, yw) = \nabla yw \notin A. \]

\[ \implies A \ni \nabla yw \notin A. \text{ Impossible!} \]

Assumption that \( \nabla x = \nabla y \) is false.
Distinguishable strings means different states: Proof

Lemma

\( L: \) regular language.

\( M = (Q, \Sigma, \delta, s, A): \) DFA for \( L. \)

If \( x, y \in \Sigma^* \) are distinguishable, then \( \nabla x \neq \nabla y. \)

Proof.

Assume for the sake of contradiction that \( \nabla x = \nabla y. \)

By assumption \( \exists w \in \Sigma^* \) such that \( \nabla xw \in A \) and \( \nabla yw \notin A. \)

\[ \nabla xw = \delta^*(s, xw) = \delta^*(\nabla x, w) = \delta^*(\nabla y, w) = \delta^*(s, yw) = \nabla yw \notin A. \]

\[ \Rightarrow A \ni \nabla yw \notin A. \] Impossible!

Assumption that \( \nabla x = \nabla y \) is false.
Distinguishable strings means different states: Proof

Lemma

\( L: \) regular language.
\( M = (Q, \Sigma, \delta, s, A): \) DFA for \( L. \)

If \( x, y \in \Sigma^* \) are distinguishable, then \( \nabla x \neq \nabla y. \)

Proof.

Assume for the sake of contradiction that \( \nabla x = \nabla y. \)

By assumption \( \exists w \in \Sigma^* \) such that \( \nabla xw \in A \) and \( \nabla yw \notin A. \)

\[ \implies A \ni \nabla xw = \delta^* (s, xw) = \delta^* (\nabla x, w) = \delta^* (\nabla y, w) = \delta^* (s, yw) = \nabla yw \notin A. \]

\[ \implies A \ni \nabla yw \notin A. \] Impossible!

Assumption that \( \nabla x = \nabla y \) is false.
Distinguishable strings means different states: Proof

Lemma

\( L \): regular language.

\( M = (Q, \Sigma, \delta, s, A) \): DFA for \( L \).

If \( x, y \in \Sigma^* \) are distinguishable, then \( \nabla x \neq \nabla y \).

Proof.

Assume for the sake of contradiction that \( \nabla x = \nabla y \).

By assumption \( \exists w \in \Sigma^* \) such that \( \nabla x w \in A \) and \( \nabla y w \notin A \).

\[ \Rightarrow A \ni \nabla x w = \delta^* (s, xw) = \delta^* (\nabla x, w) = \delta^* (\nabla y, w) \]

\[ = \delta^* (s, yw) = \nabla y w \notin A. \]

\[ \Rightarrow A \ni \nabla y w \notin A. \] Impossible!

Assumption that \( \nabla x = \nabla y \) is false.
Distinguishable strings means different states: Proof

**Lemma**

$L$: regular language.

$M = (Q, \Sigma, \delta, s, A)$: DFA for $L$.

If $x, y \in \Sigma^*$ are distinguishable, then $\nabla x \neq \nabla y$.

**Proof.**

Assume for the sake of contradiction that $\nabla x = \nabla y$.

By assumption $\exists w \in \Sigma^*$ such that $\nabla x w \in A$ and $\nabla y w \notin A$.

$\implies A \ni \nabla x w = \delta^*(s, xw) = \delta^*(\nabla x, w) = \delta^*(\nabla y, w) = \delta^*(s, yw) = \nabla y w \notin A.$

$\implies A \ni \nabla y w \notin A$. Impossible!

Assumption that $\nabla x = \nabla y$ is false.
Distinguishable strings means different states: Proof

Lemma

\( L \): regular language.

\( M = (Q, \Sigma, \delta, s, A) \): DFA for \( L \).

If \( x, y \in \Sigma^* \) are distinguishable, then \( \nabla x \neq \nabla y \).

Proof.

Assume for the sake of contradiction that \( \nabla x = \nabla y \).

By assumption \( \exists w \in \Sigma^* \) such that \( \nabla x w \in A \) and \( \nabla y w \notin A \).

\[ \implies A \ni \nabla x w = \delta^*(s, xw) = \delta^*(\nabla x, w) = \delta^*(\nabla y, w) = \delta^*(s, yw) = \nabla y w \notin A. \]

\[ \implies A \ni \nabla y w \notin A. \text{ Impossible!} \]

Assumption that \( \nabla x = \nabla y \) is false.
Distinguishable strings means different states: Proof

**Lemma**

$L$: regular language.

$M = (Q, \Sigma, \delta, s, A)$: DFA for $L$.

If $x, y \in \Sigma^*$ are distinguishable, then $\nabla x \neq \nabla y$.

**Proof.**

Assume for the sake of contradiction that $\nabla x = \nabla y$.

By assumption $\exists w \in \Sigma^*$ such that $\nabla xw \in A$ and $\nabla yw \notin A$.

\[ \implies A \ni \nabla xw = \delta^*(s, xw) = \delta^*(\nabla x, w) = \delta^*(\nabla y, w) = \delta^*(s, yw) = \nabla yw \notin A. \]

\[ \implies A \ni \nabla yw \notin A. \] Impossible!

Assumption that $\nabla x = \nabla y$ is false.
1. Prove for any \( i \neq j \) then \( 0^i \) and \( 0^j \) are distinguishable for the language \( \{0^k1^k \mid k \geq 0\} \).

2. Let \( L \) be a regular language, and let \( w_1, \ldots, w_k \) be strings that are all pairwise distinguishable for \( L \). Prove that any DFA for \( L \) must have at least \( k \) states.

3. Prove that \( \{0^k1^k \mid k \geq 0\} \) is not regular.
Review questions...

1. Prove for any \( i \neq j \) then \( 0^i \) and \( 0^j \) are distinguishable for the language \( \{0^k1^k \mid k \geq 0\} \).

2. Let \( L \) be a regular language, and let \( w_1, \ldots, w_k \) be strings that are all pairwise distinguishable for \( L \). Prove that any DFA for \( L \) must have at least \( k \) states.

3. Prove that \( \{0^k1^k \mid k \geq 0\} \) is not regular.
Review questions...

1. Prove for any $i \neq j$ then $0^i$ and $0^j$ are distinguishable for the language $\{0^k1^k \mid k \geq 0\}$.

2. Let $L$ be a regular language, and let $w_1, \ldots, w_k$ be strings that are all pairwise distinguishable for $L$. Prove that any DFA for $L$ must have at least $k$ states.

3. Prove that $\{0^k1^k \mid k \geq 0\}$ is not regular.
THE END

... 

(for now)
6.2.1

Old version: Proving non-regularity
Proof structure for showing a language $L$ is not regular:

1. For sake of contradiction, assume it is regular.
2. There exists a finite DFA $M = (Q, \Sigma, \delta, s, A)$ that accepts the language.
3. Showing that there are prefix strings $w_1, w_2, \ldots$ that are all distinguishable.
4. Define $q_i = \nabla w_i = \delta^*(s, w_i)$, for $i = 1, \ldots, \infty$.
5. $\forall i, j : i \neq j$: Since $w_i$ and $w_j$ are distinguishable $\implies q_i \neq q_j$.
6. $M$ has infinite number of states. Impossible!
7. Contradiction to $L$ being regular.
How to prove non-regularity?

Claim: Language $L$ is not regular.

Idea: Show $\#$ states in any DFA $M$ for language $L$ has infinite number of states.

Lemma

Consider three strings $x, y, w \in \Sigma^*$.

$M = (Q, \Sigma, \delta, s, A)$: DFA for language $L \subseteq \Sigma^*$.

If $\delta^*(s, xw) \in A$ and $\delta^*(s, yw) \notin A$ then $\delta^*(s, x) \neq \delta^*(s, y)$.

Proof.

Assume for the sake of contradiction that $\delta^*(s, x) = \delta^*(s, y)$.

$\implies A \ni \delta^*(s, xw) = \delta^*(\delta^*(s, x), w) = \delta^*(\delta^*(s, y), w) = \delta^*(s, yw) \notin A$

$\implies A \ni \delta^*(s, xw) \notin A$. Impossible!
How to prove non-regularity?

Claim: Language $L$ is not regular.
Idea: Show $\#$ states in any DFA $M$ for language $L$ has infinite number of states.

Lemma

Consider three strings $x, y, w \in \Sigma^*$.
$M = (Q, \Sigma, \delta, s, A)$: DFA for language $L \subseteq \Sigma^*$.
If $\delta^*(s, xw) \in A$ and $\delta^*(s, yw) \notin A$ then $\delta^*(s, x) \neq \delta^*(s, y)$.

Proof.

Assume for the sake of contradiction that $\delta^*(s, x) = \delta^*(s, y)$.

$\implies A \ni \delta^*(s, xw) = \delta^*(\delta^*(s, x), w) = \delta^*(\delta^*(s, y), w) = \delta^*(s, yw) \notin A$

$\implies A \ni \delta^*(s, xw) \notin A$. Impossible!
How to prove non-regularity?

Claim: Language $L$ is not regular.

Idea: Show # states in any DFA $M$ for language $L$ has infinite number of states.

Lemma

Consider three strings $x, y, w \in \Sigma^*$.

$M = (Q, \Sigma, \delta, s, A)$: DFA for language $L \subseteq \Sigma^*$.

If $\delta^*(s, xw) \in A$ and $\delta^*(s, yw) \notin A$ then $\delta^*(s, x) \neq \delta^*(s, y)$.

Proof.

Assume for the sake of contradiction that $\delta^*(s, x) = \delta^*(s, y)$.

$\implies$ $A \ni \delta^*(s, xw) = \delta^*(\delta^*(s, x), w) = \delta^*(\delta^*(s, y), w) = \delta^*(s, yw) \notin A$

$\implies$ $A \ni \delta^*(s, xw) \notin A$. Impossible!
How to prove non-regularity?

**Claim:** Language $L$ is not regular.

**Idea:** Show $\#$ states in any DFA $M$ for language $L$ has infinite number of states.

**Lemma**

Consider three strings $x, y, w \in \Sigma^*$.

$M = (Q, \Sigma, \delta, s, A)$: DFA for language $L \subseteq \Sigma^*$.

If $\delta^*(s, xw) \in A$ and $\delta^*(s, yw) \notin A$ then $\delta^*(s, x) \neq \delta^*(s, y)$.

**Proof.**

Assume for the sake of contradiction that $\delta^*(s, x) = \delta^*(s, y)$.

\[ \implies A \ni \delta^*(s, xw) = \delta^*(\delta^*(s, x), w) = \delta^*(\delta^*(s, y), w) = \delta^*(s, yw) \notin A \]

\[ \implies A \ni \delta^*(s, xw) \notin A. \text{ Impossible!} \]
How to prove non-regularity?

Claim: Language $L$ is not regular.

Idea: Show $\#$ states in any DFA $M$ for language $L$ has infinite number of states.

Lemma

Consider three strings $x, y, w \in \Sigma^*$.

$M = (Q, \Sigma, \delta, s, A)$: DFA for language $L \subseteq \Sigma^*$.

If $\delta^*(s, xw) \in A$ and $\delta^*(s, yw) \not\in A$ then $\delta^*(s, x) \neq \delta^*(s, y)$.

Proof.

Assume for the sake of contradiction that $\delta^*(s, x) = \delta^*(s, y)$.

$\Rightarrow A \ni \delta^*(s, xw) = \delta^*(\delta^*(s, x), w) = \delta^*(\delta^*(s, y), w) = \delta^*(s, yw) \not\in A$

$\Rightarrow A \ni \delta^*(s, xw) \not\in A$. Impossible!
How to prove non-regularity?

Claim: Language $L$ is not regular.
Idea: Show $\#$ states in any DFA $M$ for language $L$ has infinite number of states.

Lemma

Consider three strings $x, y, w \in \Sigma^*$.

$M = (Q, \Sigma, \delta, s, A)$: DFA for language $L \subseteq \Sigma^*$.

If $\delta^*(s, xw) \in A$ and $\delta^*(s, yw) \notin A$ then $\delta^*(s, x) \neq \delta^*(s, y)$.

Proof.

Assume for the sake of contradiction that $\delta^*(s, x) = \delta^*(s, y)$.

$\implies A \ni \delta^*(s, xw) = \delta^*(\delta^*(s, x), w) = \delta^*(\delta^*(s, y), w) = \delta^*(s, yw) \notin A$

$\implies A \ni \delta^*(s, xw) \notin A$. Impossible!
How to prove non-regularity?

Claim: Language $L$ is not regular.
Idea: Show $\#$ states in any DFA $M$ for language $L$ has infinite number of states.

Lemma

Consider three strings $x, y, w \in \Sigma^*$. 
$M = (Q, \Sigma, \delta, s, A)$: DFA for language $L \subseteq \Sigma^*$.
If $\delta^*(s, xw) \in A$ and $\delta^*(s, yw) \notin A$ then $\delta^*(s, x) \neq \delta^*(s, y)$.

Proof.

Assume for the sake of contradiction that $\delta^*(s, x) = \delta^*(s, y)$.

$\implies A \ni \delta^*(s, xw) = \delta^*(\delta^*(s, x), w) = \delta^*(\delta^*(s, y), w) = \delta^*(s, yw) \notin A$

$\implies A \ni \delta^*(s, xw) \notin A$. Impossible!
How to prove non-regularity?

Claim: Language $L$ is not regular.

Idea: Show that states in any DFA $M$ for language $L$ has infinite number of states.

Lemma

Consider three strings $x, y, w \in \Sigma^*$.

$M = (Q, \Sigma, \delta, s, A)$: DFA for language $L \subseteq \Sigma^*$.

If $\delta^*(s, xw) \in A$ and $\delta^*(s, yw) \notin A$ then $\delta^*(s, x) \neq \delta^*(s, y)$.

Proof.

Assume for the sake of contradiction that $\delta^*(s, x) = \delta^*(s, y)$.

$\implies A \ni \delta^*(s, xw) = \delta^*(\delta^*(s, x), w) = \delta^*(\delta^*(s, y), w) = \delta^*(s, yw) \notin A$

$\implies A \ni \delta^*(s, xw) \notin A$. Impossible!
Generalizing the argument

**Definition**

For a language $L$ over $\Sigma$ and two strings $x, y \in \Sigma^*$, $x$ and $y$ are distinguishable with respect to $L$ if there is a string $w \in \Sigma^*$ such that exactly one of $xw, yw$ is in $L$. $x, y$ are indistinguishable with respect to $L$ if there is no such $w$.

**Example:** If $i \neq j$, $0^i$ and $0^j$ are distinguishable with respect to $L = \{0^k1^k \mid k \geq 0\}$.

**Example:** $000$ and $0000$ are indistinguishable with respect to the language $L = \{w \mid w$ has $00$ as a substring$\}$. 
Generalizing the argument

**Definition**

For a language $L$ over $\Sigma$ and two strings $x, y \in \Sigma^*$, $x$ and $y$ are **distinguishable** with respect to $L$ if there is a string $w \in \Sigma^*$ such that exactly one of $xw, yw$ is in $L$. $x, y$ are **indistinguishable** with respect to $L$ if there is no such $w$.

**Example:** If $i \neq j$, $0^i$ and $0^j$ are distinguishable with respect to $L = \{0^k1^k \mid k \geq 0\}$.

**Example:** $000$ and $0000$ are indistinguishable with respect to the language $L = \{w \mid w$ has $00$ as a substring$\}$. 
Generalizing the argument

**Definition**

For a language \( L \) over \( \Sigma \) and two strings \( x, y \in \Sigma^* \), \( x \) and \( y \) are **distinguishable** with respect to \( L \) if there is a string \( w \in \Sigma^* \) such that exactly one of \( xw,yw \) is in \( L \).

\( x, y \) are **indistinguishable** with respect to \( L \) if there is no such \( w \).

**Example:** If \( i \neq j \), \( 0^i \) and \( 0^j \) are distinguishable with respect to \( L = \{0^k1^k \mid k \geq 0\} \).

**Example:** 000 and 0000 are indistinguishable with respect to the language \( L = \{w \mid w \text{ has 00 as a substring}\} \).
Generalizing the argument

Definition

For a language $L$ over $\Sigma$ and two strings $x, y \in \Sigma^*$, $x$ and $y$ are distinguishable with respect to $L$ if there is a string $w \in \Sigma^*$ such that exactly one of $xw, yw$ is in $L$. $x, y$ are indistinguishable with respect to $L$ if there is no such $w$.

Example: If $i \neq j$, $0^i$ and $0^j$ are distinguishable with respect to $L = \{0^k1^k \mid k \geq 0\}$

Example: 000 and 0000 are indistinguishable with respect to the language $L = \{w \mid w$ has 00 as a substring$\}$. 
Wee Lemma

Lemma

Suppose $L = L(M)$ for some DFA $M = (Q, \Sigma, \delta, s, A)$ and suppose $x, y$ are distinguishable with respect to $L$. Then $\delta^*(s, x) \neq \delta^*(s, y)$.

Proof.

Since $x, y$ are distinguishable let $w$ be the distinguishing suffix. If $\delta^*(s, x) = \delta^*(s, y)$ then $M$ will either accept both the strings $xw, yw$, or reject both. But exactly one of them is in $L$, a contradiction.
Wee Lemma

Lemma

Suppose \( L = L(M) \) for some DFA \( M = (Q, \Sigma, \delta, s, A) \) and suppose \( x, y \) are distinguishable with respect to \( L \). Then \( \delta^*(s, x) \neq \delta^*(s, y) \).

Proof.

Since \( x, y \) are distinguishable let \( w \) be the distinguishing suffix. If \( \delta^*(s, x) = \delta^*(s, y) \) then \( M \) will either accept both the strings \( xw, yw \), or reject both. But exactly one of them is in \( L \), a contradiction.
THE END
...
(for now)
6.3

Fooling sets: Proving non-regularity
Fooling Sets

**Definition**
For a language $L$ over $\Sigma$ a set of strings $F$ (could be infinite) is a fooling set or distinguishing set for $L$ if every two distinct strings $x, y \in F$ are distinguishable.

Example: $F = \{0^i \mid i \geq 0\}$ is a fooling set for the language $L = \{0^k1^k \mid k \geq 0\}$.

**Theorem**
Suppose $F$ is a fooling set for $L$. If $F$ is finite then there is no DFA $M$ that accepts $L$ with less than $|F|$ states.
Fooling Sets

**Definition**
For a language $L$ over $\Sigma$ a set of strings $F$ (could be infinite) is a fooling set or distinguishing set for $L$ if every two distinct strings $x, y \in F$ are distinguishable.

**Example:** $F = \{0^i \mid i \geq 0\}$ is a fooling set for the language $L = \{0^k1^k \mid k \geq 0\}$.

**Theorem**
Suppose $F$ is a fooling set for $L$. If $F$ is finite then there is no DFA $M$ that accepts $L$ with less than $|F|$ states.
Fooling Sets

Definition
For a language \( L \) over \( \Sigma \) a set of strings \( F \) (could be infinite) is a fooling set or distinguishing set for \( L \) if every two distinct strings \( x, y \in F \) are distinguishable.

Example: \( F = \{0^i \mid i \geq 0\} \) is a fooling set for the language \( L = \{0^k1^k \mid k \geq 0\} \).

Theorem
Suppose \( F \) is a fooling set for \( L \). If \( F \) is finite then there is no DFA \( M \) that accepts \( L \) with less than \( |F| \) states.
Recall

Already proved the following lemma:

**Lemma**

\( L: \) regular language.
\( M = (Q, \Sigma, \delta, s, A): \) DFA for \( L. \)

If \( x, y \in \Sigma^* \) are distinguishable, then \( \nabla x \neq \nabla y. \)

Reminder: \( \nabla x = \delta^*(s, x). \)
Proof of theorem

Theorem (Reworded.)

**L**: A language  
**F**: a fooling set for **L**.

If **F** is finite then any **DFA** **M** that accepts **L** has at least |**F**| states.

Proof.

Let **F** = \{\(w_1, w_2, \ldots, w_m\)\} be the fooling set.

Let **M** = (\(Q, \Sigma, \delta, s, A\)) be any **DFA** that accepts **L**.

Let \(q_i = \nabla w_i = \delta^*(s, x_i)\).

By lemma \(q_i \neq q_j\) for all \(i \neq j\).

As such, |\(Q\)| ≥ |\(\{q_1, \ldots, q_m\}\)| = |\(\{w_1, \ldots, w_m\}\)| = |**F**|.
Proof of theorem

Theorem (Reworded.)

$L$: A language
$F$: a fooling set for $L$.

If $F$ is finite then any DFA $M$ that accepts $L$ has at least $|F|$ states.

Proof.

Let $F = \{w_1, w_2, \ldots, w_m\}$ be the fooling set.
Let $M = (Q, \Sigma, \delta, s, A)$ be any DFA that accepts $L$.
Let $q_i = \nabla w_i = \delta^*(s, x_i)$.

By lemma $q_i \neq q_j$ for all $i \neq j$.

As such, $|Q| \geq |\{q_1, \ldots, q_m\}| = |\{w_1, \ldots, w_m\}| = |F|$.
Theorem (Reworded.)

Let $L$ be a language and $F$ be a fooling set for $L$.

If $F$ is finite then any DFA $M$ that accepts $L$ has at least $|F|$ states.

Proof.

Let $F = \{w_1, w_2, \ldots, w_m\}$ be the fooling set.

Let $M = (Q, \Sigma, \delta, s, A)$ be any DFA that accepts $L$.

Let $q_i = \bigtriangledown w_i = \delta^*(s, x_i)$.

By lemma $q_i \neq q_j$ for all $i \neq j$.

As such, $|Q| \geq |\{q_1, \ldots, q_m\}| = |\{w_1, \ldots, w_m\}| = |F|$. □
Corollary

If $L$ has an infinite fooling set $F$ then $L$ is not regular.

Proof.

Let $w_1, w_2, \ldots \subseteq F$ be an infinite sequence of strings such that every pair of them are distinguishable.

Assume for contradiction that $\exists M$ a DFA for $L$.

Let $F_i = \{w_1, \ldots, w_i\}$.

By theorem, $\# \text{ states of } M \geq |F_i| = i$, for all $i$.

As such, number of states in $M$ is infinite.

Contradiction: DFA = deterministic finite automata. But $M$ not finite.
Infinite Fooling Sets

Corollary

If $L$ has an infinite fooling set $F$ then $L$ is not regular.

Proof.

Let $w_1, w_2, \ldots \subseteq F$ be an infinite sequence of strings such that every pair of them are distinguishable.

Assume for contradiction that $\exists M$ a DFA for $L$.

Let $F_i = \{w_1, \ldots, w_i\}$.

By theorem, $\# \text{ states of } M \geq |F_i| = i$, for all $i$.

As such, number of states in $M$ is infinite.

Contradiction: $\text{DFA} = \text{deterministic finite automata}$. But $M$ not finite.
Corollary

If $L$ has an infinite fooling set $F$ then $L$ is not regular.

Proof.

Let $w_1, w_2, \ldots \subseteq F$ be an infinite sequence of strings such that every pair of them are distinguishable.

Assume for contradiction that $\exists M$ a DFA for $L$.

Let $F_i = \{w_1, \ldots, w_i\}$.

By theorem, $\# \text{ states of } M \geq |F_i| = i$, for all $i$.

As such, number of states in $M$ is infinite.

Contradiction: DFA = deterministic finite automata. But $M$ not finite.
Examples

- \( \{0^k1^k \mid k \geq 0\} \)
- \{bitstrings with equal number of 0s and 1s\}
- \( \{0^k1^\ell \mid k \neq \ell\} \)
Examples

- $\{0^k1^k \mid k \geq 0\}$
- \{bitstrings with equal number of 0s and 1s\}
- $\{0^k1^\ell \mid k \neq \ell\}$
Examples

- $\{0^k1^k \mid k \geq 0\}$
- \{bitstrings with equal number of 0s and 1s\}
- $\{0^k1^\ell \mid k \neq \ell\}$
Harder example: The language of squares is not regular

\[ \{0^k | k \geq 0\} \]
Really hard: Primes are not regular

An exercise left for your enjoyment

\[ \{0^k \mid k \text{ is a prime number} \} \]

Hints:

1. Probably easier to prove directly on the automata.
2. There are infinite number of prime numbers.
3. For every \( n > 0 \), observe that \( n!, n! + 1, \ldots, n! + n \) are all composite – there are arbitrarily big gaps between prime numbers.
THE END

... (for now)
6.3.1

Exponential gap in number of states between DFA and NFA sizes
Exponential gap between NFA and DFA size

\[ L_4 = \{ w \in \{0, 1\}^* \mid w \text{ has a } 1 \text{ located } 4 \text{ positions from the end}\} \]
Exponential gap between NFA and DFA size

$L_k = \{w \in \{0, 1\}^* \mid w \text{ has a } 1 \text{ } k \text{ positions from the end}\}$

Recall that $L_k$ is accepted by a NFA $N$ with $k + 1$ states.

**Theorem**

Every DFA that accepts $L_k$ has at least $2^k$ states.

**Claim**

$F = \{w \in \{0, 1\}^* : |w| = k\}$ is a fooling set of size $2^k$ for $L_k$.

**Why?**

- Suppose $a_1a_2\ldots a_k$ and $b_1b_2\ldots b_k$ are two distinct bitstrings of length $k$
- Let $i$ be first index where $a_i \neq b_i$
- $y = 0^{k-i-1}$ is a distinguishing suffix for the two strings
**Exponential gap between NFA and DFA size**

$L_k = \{ w \in \{0, 1\}^* \mid w \text{ has a } 1 \text{ } k \text{ positions from the end}\}$

Recall that $L_k$ is accepted by a NFA $N$ with $k + 1$ states.

### Theorem

Every DFA that accepts $L_k$ has at least $2^k$ states.

### Claim

$F = \{ w \in \{0, 1\}^* : |w| = k \}$ is a fooling set of size $2^k$ for $L_k$.

**Why?**

- Suppose $a_1a_2\ldots a_k$ and $b_1b_2\ldots b_k$ are two distinct bitstrings of length $k$
- Let $i$ be first index where $a_i \neq b_i$
- $y = 0^{k-i-1}$ is a distinguishing suffix for the two strings
Exponential gap between NFA and DFA size

$L_k = \{ w \in \{0, 1\}^* | w \text{ has a 1 \(k\) positions from the end}\}$

Recall that $L_k$ is accepted by a NFA $N$ with $k + 1$ states.

**Theorem**

*Every DFA that accepts $L_k$ has at least $2^k$ states.*

**Claim**

$F = \{ w \in \{0, 1\}^* : |w| = k \}$ is a fooling set of size $2^k$ for $L_k$.

Why?

- Suppose $a_1a_2\ldots a_k$ and $b_1b_2\ldots b_k$ are two distinct bitstrings of length $k$
- Let $i$ be first index where $a_i \neq b_i$
- $y = 0^{k-i-1}$ is a distinguishing suffix for the two strings
Exponential gap between NFA and DFA size

\[ L_k = \{ w \in \{0, 1\}^* \mid w \text{ has a } 1 \text{ } k \text{ positions from the end} \} \]

Recall that \( L_k \) is accepted by a NFA \( N \) with \( k + 1 \) states.

**Theorem**

*Every DFA that accepts \( L_k \) has at least \( 2^k \) states.*

**Claim**

\[ F = \{ w \in \{0, 1\}^* : |w| = k \} \] is a fooling set of size \( 2^k \) for \( L_k \).

**Why?**

- Suppose \( a_1 a_2 \ldots a_k \) and \( b_1 b_2 \ldots b_k \) are two distinct bitstrings of length \( k \)
- Let \( i \) be first index where \( a_i \neq b_i \)
- \( y = 0^{k-i-1} \) is a distinguishing suffix for the two strings
Exponential gap between NFA and DFA size

\[ L_k = \{ w \in \{0, 1\}^* \mid w \text{ has a 1 } k \text{ positions from the end} \} \]
Recall that \( L_k \) is accepted by a NFA \( N \) with \( k + 1 \) states.

**Theorem**

Every DFA that accepts \( L_k \) has at least \( 2^k \) states.

**Claim**

\( F = \{ w \in \{0, 1\}^* : |w| = k \} \) is a fooling set of size \( 2^k \) for \( L_k \).

Why?

- Suppose \( a_1a_2 \ldots a_k \) and \( b_1b_2 \ldots b_k \) are two distinct bitstrings of length \( k \)
- Let \( i \) be first index where \( a_i \neq b_i \)
- \( y = 0^{k-i-1} \) is a distinguishing suffix for the two strings
How do we pick a fooling set $F$?

- If $x, y$ are in $F$ and $x \neq y$ they should be distinguishable! Of course.

- All strings in $F$ except maybe one should be prefixes of strings in the language $L$. For example if $L = \{0^k1^k \mid k \geq 0\}$ do not pick 1 and 10 (say). Why?
THE END

...(for now)
6.4 Closure properties: Proving non-regularity
Non-regularity via closure properties

\[ H = \{ \text{bitstrings with equal number of 0s and 1s} \} \]

\[ H' = \{ 0^k1^k \mid k \geq 0 \} \]

Suppose we have already shown that \( L' \) is non-regular. Can we show that \( L \) is non-regular without using the fooling set argument from scratch?

\[ H' = H \cap L(0^*1^*) \]

Claim: The above and the fact that \( L' \) is non-regular implies \( L \) is non-regular. Why?

Suppose \( H \) is regular. Then since \( L(0^*1^*) \) is regular, and regular languages are closed under intersection, \( H' \) also would be regular. But we know \( H' \) is not regular, a contradiction.
Non-regularity via closure properties

\[ H = \{ \text{bitstrings with equal number of 0s and 1s} \} \]

\[ H' = \{ 0^k 1^k \mid k \geq 0 \} \]

Suppose we have already shown that \( L' \) is non-regular. Can we show that \( L \) is non-regular without using the fooling set argument from scratch?

\[ H' = H \cap L(0^*1^*) \]

Claim: The above and the fact that \( L' \) is non-regular implies \( L \) is non-regular. Why?

Suppose \( H \) is regular. Then since \( L(0^*1^*) \) is regular, and regular languages are closed under intersection, \( H' \) also would be regular. But we know \( H' \) is not regular, a contradiction.
Non-regularity via closure properties

\[ H = \{ \text{bitstrings with equal number of 0s and 1s} \} \]

\[ H' = \{ 0^k1^k \mid k \geq 0 \} \]

Suppose we have already shown that \( L' \) is non-regular. Can we show that \( L \) is non-regular without using the fooling set argument from scratch?

\[ H' = H \cap L(0^*1^*) \]

**Claim:** The above and the fact that \( L' \) is non-regular implies \( L \) is non-regular. Why?

Suppose \( H \) is regular. Then since \( L(0^*1^*) \) is regular, and regular languages are closed under intersection, \( H' \) also would be regular. But we know \( H' \) is not regular, a contradiction.
Non-regularity via closure properties

General recipe:

Apply closure properties

L_1
L_2
L_n
L_non-regular

KNOWN REGULAR
UNKNOWN

Har-Peled (UIUC) CS374 Fall 2020 39 / 59
Proving non-regularity: Summary

- Method of distinguishing suffixes. To prove that $L$ is non-regular find an infinite fooling set.
- Closure properties. Use existing non-regular languages and regular languages to prove that some new language is non-regular.
- Pumping lemma. We did not cover it but it is sometimes an easier proof technique to apply, but not as general as the fooling set technique.
THE END

... 

(for now)
6.5
Myhill-Nerode Theorem
“Myhill-Nerode Theorem”: A regular language $L$ has a unique (up to naming) minimal automata, and it can be computed efficiently once any DFA is given for $L$. 
6.5.1
Myhill-Nerode Theorem: Equivalence between strings
Indistinguishability

Recall:

Definition

For a language $L$ over $\Sigma$ and two strings $x, y \in \Sigma^*$ we say that $x$ and $y$ are distinguishable with respect to $L$ if there is a string $w \in \Sigma^*$ such that exactly one of $xw, yw$ is in $L$. $x, y$ are indistinguishable with respect to $L$ if there is no such $w$.

Given language $L$ over $\Sigma$ define a relation $\equiv_L$ over strings in $\Sigma^*$ as follows: $x \equiv_L y$ iff $x$ and $y$ are indistinguishable with respect to $L$.

Definition

$x \equiv_L y$ means that $\forall w \in \Sigma^*: xw \in L \iff yw \in L$.

In words: $x$ is equivalent to $y$ under $L$. 
Indistinguishability

Recall:

**Definition**

For a language $L$ over $\Sigma$ and two strings $x, y \in \Sigma^*$ we say that $x$ and $y$ are distinguishable with respect to $L$ if there is a string $w \in \Sigma^*$ such that exactly one of $xw, yw$ is in $L$. $x, y$ are indistinguishable with respect to $L$ if there is no such $w$.

Given language $L$ over $\Sigma$ define a relation $\equiv_L$ over strings in $\Sigma^*$ as follows: $x \equiv_L y$ iff $x$ and $y$ are indistinguishable with respect to $L$.

**Definition**

$x \equiv_L y$ means that $\forall w \in \Sigma^*: xw \in L \iff yw \in L$.

In words: $x$ is equivalent to $y$ under $L$. 
Example: Equivalence classes
Claim

\( L \) is an equivalence relation over \( \Sigma^* \).

Proof.

1. **Reflexive:** \( \forall x \in \Sigma^*: \forall w \in \Sigma^*: xw \in L \iff xw \in L \rightarrow x \equiv_L x \).

2. **Symmetry:** \( x \equiv_L y \) then \( \forall w \in \Sigma^*: xw \in L \iff yw \in L \forall w \in \Sigma^*: yw \in L \iff xw \in L \Rightarrow y \equiv_L x \).

3. **Transitivity:** \( x \equiv_L y \) and \( y \equiv_L z \)

\( \forall w \in \Sigma^*: xw \in L \iff yw \in L \) and \( \forall w \in \Sigma^*: yw \in L \iff zw \in L \)

\( \Rightarrow \forall w \in \Sigma^*: xw \in L \iff zw \in L \Rightarrow x \equiv_L z \).
Indistinguishability

Claim

\[ \equiv_L \text{ is an equivalence relation over } \Sigma^*. \]

Proof.

1. Reflexive: \( \forall x \in \Sigma^*: \forall w \in \Sigma^*: xw \in L \iff xw \in L. \Rightarrow x \equiv_L x. \)

2. Symmetry: \( x \equiv_L y \) then \( \forall w \in \Sigma^*: xw \in L \iff yw \in L \)

3. Transitivity: \( x \equiv_L y \) and \( y \equiv_L z \)

\[ \forall w \in \Sigma^*: xw \in L \iff yw \in L \text{ and } \forall w \in \Sigma^*: yw \in L \iff zw \in L \]

\[ \Rightarrow \forall w \in \Sigma^*: xw \in L \iff zw \in L \]

\[ \Rightarrow x \equiv_L z. \]
Indistinguishability

Claim

\[ \equiv_L \text{ is an equivalence relation over } \Sigma^*. \]

Proof.

1. Reflexive: \( \forall x \in \Sigma^*: \forall w \in \Sigma^*: xw \in L \iff xw \in L. \implies x \equiv_L x. \)

2. Symmetry: \( x \equiv_L y \) then \( \forall w \in \Sigma^*: xw \in L \iff yw \in L \)
   \( \forall w \in \Sigma^*: yw \in L \iff xw \in L \implies y \equiv_L x. \)

3. Transitivity: \( x \equiv_L y \) and \( y \equiv_L z \)
   \( \forall w \in \Sigma^*: xw \in L \iff yw \in L \) and \( \forall w \in \Sigma^*: yw \in L \iff zw \in L \)
   \( \implies \forall w \in \Sigma^*: xw \in L \iff zw \in L \)
   \( \implies x \equiv_L z. \)
Claim

$\equiv_L$ is an equivalence relation over $\Sigma^*$.  

Proof.

1. Reflexive: $\forall x \in \Sigma^* : \forall w \in \Sigma^* : xw \in L \iff xw \in L$. $\implies x \equiv_L x$.

2. Symmetry: $x \equiv_L y$ then $\forall w \in \Sigma^*: xw \in L \iff yw \in L$ and $\forall w \in \Sigma^* : yw \in L \iff xw \in L$. $\implies y \equiv_L x$.

3. Transitivity: $x \equiv_L y$ and $y \equiv_L z$ $\forall w \in \Sigma^* : xw \in L \iff yw \in L$ and $\forall w \in \Sigma^* : yw \in L \iff zw \in L$ $\implies \forall w \in \Sigma^* : xw \in L \iff zw \in L$ $\implies x \equiv_L z$. 
Claim

$\equiv_L$ is an equivalence relation over $\Sigma^*$.

Proof.

1. Reflexive: $\forall x \in \Sigma^*$: $\forall w \in \Sigma^*$: $xw \in L \iff xw \in L \implies x \equiv_L x$.

2. Symmetry: $x \equiv_L y$ then $\forall w \in \Sigma^*$: $xw \in L \iff yw \in L$

3. Transitivity: $x \equiv_L y$ and $y \equiv_L z$

   $\forall w \in \Sigma^*$: $xw \in L \iff yw \in L$ and $\forall w \in \Sigma^*$: $yw \in L \iff zw \in L$

   $\implies \forall w \in \Sigma^*$: $xw \in L \iff zw \in L$

   $\implies x \equiv_L z$. 
Indistinguishability

Claim

\[ \equiv_L \text{ is an equivalence relation over } \Sigma^*. \]

Proof.

1. Reflexive: \( \forall x \in \Sigma^*: \forall w \in \Sigma^*: xw \in L \iff xw \in L. \iff x \equiv_L x. \)

2. Symmetry: \( x \equiv_L y \) then \( \forall w \in \Sigma^*: xw \in L \iff yw \in L \)

3. Transitivity: \( x \equiv_L y \) and \( y \equiv_L z \)

\( \forall w \in \Sigma^*: xw \in L \iff yw \in L \) and \( \forall w \in \Sigma^*: yw \in L \iff zw \in L \)

\[ \iff \forall w \in \Sigma^*: xw \in L \iff zw \in L \iff x \equiv_L z. \]
Indistinguishability

Claim

$\equiv_L$ is an equivalence relation over $\Sigma^*$.

Proof.

1. Reflexive: $\forall x \in \Sigma^*: \forall w \in \Sigma^*: xw \in L \iff xw \in L \implies x \equiv_L x$.

2. Symmetry: $x \equiv_L y$ then $\forall w \in \Sigma^*: xw \in L \iff yw \in L$.

3. Transitivity: $x \equiv_L y$ and $y \equiv_L z$

   $\forall w \in \Sigma^*: xw \in L \iff yw \in L$ and $\forall w \in \Sigma^*: yw \in L \iff zw \in L$

   $\implies \forall w \in \Sigma^*: xw \in L \iff zw \in L$

   $\implies x \equiv_L z$. 
Equivalences over automatas...

Claim (Just proved.)

\[ \equiv_L \text{ is an equivalence relation over } \Sigma^*. \]

Therefore, \( \equiv_L \) partitions \( \Sigma^* \) into a collection of equivalence classes.

Definition

\( L \): A language For a string \( x \in \Sigma^* \), let

\[ [x] = [x]_L = \{ y \in \Sigma^* | x \equiv_L y \} \]

be the equivalence class of \( x \) according to \( L \).

Definition

\( [L] = \{ [x]_L | x \in \Sigma^* \} \) is the set of equivalence classes of \( L \).
Strings in the same equivalence class are indistinguishable

**Lemma**

Let $x, y$ be two distinct strings.

$x \equiv_L y \iff x, y$ are indistinguishable for $L$.

**Proof.**

$x \equiv_L y \iff \forall w \in \Sigma^*: xw \in L \iff yw \in L$

$x$ and $y$ are indistinguishable for $L$.

$x \not\equiv_L y \iff \exists w \in \Sigma^*: xw \in L$ and $yw \not\in L$

$\iff x$ and $y$ are distinguishable for $L$. 

Strings in the same equivalence class are indistinguishable

**Lemma**

Let \( x, y \) be two distinct strings.

\[ x \equiv_L y \iff x, y \text{ are indistinguishable for } L. \]

**Proof.**

\[ x \equiv_L y \iff \forall w \in \Sigma^*: xw \in L \iff yw \in L \]

\( x \) and \( y \) are indistinguishable for \( L \).

\[ x \not\equiv_L y \iff \exists w \in \Sigma^*: xw \in L \text{ and } yw \not\in L \]

\( x \) and \( y \) are distinguishable for \( L \).
Strings in the same equivalence class are indistinguishable

**Lemma**

Let $x, y$ be two distinct strings. 

$x \equiv_L y \iff x, y$ are indistinguishable for $L$.

**Proof.**

$x \equiv_L y \implies \forall w \in \Sigma^*: xw \in L \iff yw \in L$

$x$ and $y$ are indistinguishable for $L$.

$x \not\equiv_L y \implies \exists w \in \Sigma^*: xw \in L$ and $yw \not\in L$

$\implies x$ and $y$ are distinguishable for $L$. 

Har-Peled (UIUC)  
CS374  
Fall 2020  
49 / 59
Strings in the same equivalence class are indistinguishable

Lemma

Let $x, y$ be two distinct strings.

$x \equiv_L y \iff x, y$ are indistinguishable for $L$.

Proof.

$x \equiv_L y \implies \forall w \in \Sigma^*: xw \in L \iff yw \in L$

$x$ and $y$ are indistinguishable for $L$.

$x \not\equiv_L y \implies \exists w \in \Sigma^*: xw \in L$ and $yw \not\in L$

$x$ and $y$ are distinguishable for $L$. 
Strings in the same equivalence class are indistinguishable

**Lemma**

Let \( x, y \) be two distinct strings.

\[ x \equiv_L y \iff x, y \text{ are indistinguishable for } L. \]

**Proof.**

\[ x \equiv_L y \implies \forall w \in \Sigma^*: xw \in L \iff yw \in L \]

\( x \) and \( y \) are indistinguishable for \( L \).

\[ x \not\equiv_L y \implies \exists w \in \Sigma^*: xw \in L \text{ and } yw \not\in L \]

\[ \implies x \text{ and } y \text{ are distinguishable for } L. \]
All strings arriving at a state are in the same class

**Lemma**

$M = (Q, \Sigma, \delta, s, A)$ a DFA for a language $L$.

For any $q \in A$, let $L_q = \{ w \in \Sigma^* | \nabla w = \delta^*(s, w) = q \}$.

Then, there exists a string $x$, such that $L_q \subseteq [x]_L$. 

Har-Peled (UIUC)
An inefficient automata
THE END

... 

(for now)
6.5.2
Stating and proving the Myhill-Nerode Theorem
Equivalences over automatas...

**Claim (Just proved)**

*Let* \( x, y \) *be two distinct strings.*

\[ x \equiv_L y \iff x, y \text{ are indistinguishable for } L. \]

**Corollary**

*If* \( \equiv_L \) *is finite with* \( n \) *equivalence classes then there is a fooling set* \( F \) *of size* \( n \) *for* \( L \).  

**Corollary**

*If* \( \equiv_L \) *has infinite number of equivalence classes* \( \implies \exists \) *infinite fooling set for* \( L \).  

\( \implies L \text{ is not regular.} \)
Claim (Just proved)

Let $x, y$ be two distinct strings.

$x \equiv_L y \iff x, y$ are indistinguishable for $L$.

Corollary

If $\equiv_L$ is finite with $n$ equivalence classes then there is a fooling set $F$ of size $n$ for $L$.

Corollary

If $\equiv_L$ has infinite number of equivalence classes $\implies \exists$ infinite fooling set for $L$.

$\implies L$ is not regular.
Claim (Just proved)

Let $x, y$ be two distinct strings.

$x \equiv_L y \iff x, y$ are indistinguishable for $L$.

Corollary

If $\equiv_L$ is finite with $n$ equivalence classes then there is a fooling set $F$ of size $n$ for $L$.

Corollary

If $\equiv_L$ has infinite number of equivalence classes $\implies \exists$ infinite fooling set for $L$.

$\implies L$ is not regular.
Equivalence classes as automata

Lemma

For all $x, y \in \Sigma^*$, if $[x]_L = [y]_L$, then for any $a \in \Sigma$, we have $[xa]_L = [ya]_L$.

Proof.

$[x] = [y] \implies \forall w \in \Sigma^*: xw \in L \iff yw \in L$

$\implies \forall w' \in \Sigma^*: xaw' \in L \iff yaw' \in L$

$\implies [xa]_L = [ya]_L$.
Equivalence classes as automata

Lemma

For all $x, y \in \Sigma^*$, if $[x]_L = [y]_L$, then for any $a \in \Sigma$, we have $[xa]_L = [ya]_L$. 

Proof.

$[x] = [y] \iff \forall w \in \Sigma^*: xw \in L \iff yw \in L$

$\iff \forall w' \in \Sigma^*: xaw' \in L \iff yaw' \in L \quad // w = aw'$

$\iff [xa]_L = [ya]_L$. 

\[\square\]
Lemma

For all \( x, y \in \Sigma^* \), if \( [x]_L = [y]_L \), then for any \( a \in \Sigma \), we have \( [xa]_L = [ya]_L \).

Proof.

\[
[x] = [y] \implies \forall w \in \Sigma^*: xw \in L \iff yw \in L \\
\implies \forall w' \in \Sigma^*: xaw' \in L \iff yaw' \in L \\
\implies [xa]_L = [ya]_L.
\]
Lemma

For all \( x, y \in \Sigma^* \), if \([x]_L = [y]_L\), then for any \( a \in \Sigma \), we have \([xa]_L = [ya]_L\).

Proof.

\[ [x] = [y] \implies \forall w \in \Sigma^*: \ xw \in L \iff yw \in L \]
\[ \implies \forall w' \in \Sigma^*: \ xaw' \in L \iff yaw' \in L \quad // \ w = aw' \]
\[ \implies [xa]_L = [ya]_L. \]
Lemma

If $L$ has $n$ distinct equivalence classes, then there is a DFA that accepts it using $n$ states.

Proof.

Set of states: $Q = [L]$
Start state: $s = [\varepsilon L]$
Accept states: $A = \{[x]_L \mid x \in L\}$
Transition function: $\delta([x]_L, a) = [xa]_L$

$M = (Q, \Sigma, \delta, s, A)$: The resulting DFA.
Clearly, $M$ is a DFA with $n$ states, and it accepts $L$.  

Har-Peled (UIUC)  
CS374  
Fall 2020
Lemma

If \( L \) has \( n \) distinct equivalence classes, then there is a DFA that accepts it using \( n \) states.

Proof.

Set of states: \( Q = [L] \)
Start state: \( s = [\varepsilon]_L \).
Accept states: \( A = \{[x]_L \mid x \in L\} \).
Transition function: \( \delta([x]_L, a) = [xa]_L \).
\( M = (Q, \Sigma, \delta, s, A) \): The resulting DFA.
Clearly, \( M \) is a DFA with \( n \) states, and it accepts \( L \).
Lemma

If \( L \) has \( n \) distinct equivalence classes, then there is a DFA that accepts it using \( n \) states.

Proof.

Set of states: \( Q = [L] \)
Start state: \( s = [\varepsilon]_L \).
Accept states: \( A = \{ [x]_L \mid x \in L \} \).
Transition function: \( \delta([x]_L, a) = [xa]_L \).
\( M = (Q, \Sigma, \delta, s, A) \): The resulting DFA.
Clearly, \( M \) is a DFA with \( n \) states, and it accepts \( L \).
Set of equivalence classes

Lemma

If $L$ has $n$ distinct equivalence classes, then there is a DFA that accepts it using $n$ states.

Proof.

Set of states: $Q = [L]$
Start state: $s = [\varepsilon]_L$.
Accept states: $A = \{[x]_L \mid x \in L\}$.
Transition function: $\delta([x]_L, a) = [xa]_L$.
$M = (Q, \Sigma, \delta, s, A)$: The resulting DFA.
Clearly, $M$ is a DFA with $n$ states, and it accepts $L$. 

\[\square\]
Lemma

If $L$ has $n$ distinct equivalence classes, then there is a DFA that accepts it using $n$ states.

Proof.

Set of states: $Q = [L]$
Start state: $s = [\varepsilon]_L$.
Accept states: $A = \{[x]_L \mid x \in L\}$.
Transition function: $\delta([x]_L, a) = [xa]_L$.

$M = (Q, \Sigma, \delta, s, A)$: The resulting DFA.

Clearly, $M$ is a DFA with $n$ states, and it accepts $L$. 
Lemma

If \( L \) has \( n \) distinct equivalence classes, then there is a DFA that accepts it using \( n \) states.

Proof.

Set of states: \( Q = [L] \)

Start state: \( s = [\varepsilon]_L \).

Accept states: \( A = \{[x]_L \mid x \in L\} \).

Transition function: \( \delta([x]_L, a) = [xa]_L \).

\( M = (Q, \Sigma, \delta, s, A) \): The resulting DFA.

Clearly, \( M \) is a DFA with \( n \) states, and it accepts \( L \).
Myhill-Nerode Theorem

**Theorem (Myhill-Nerode)**

\[ L \text{ is regular } \iff \equiv_L \text{ has a finite number of equivalence classes.} \]

If \( \equiv_L \) is finite with \( n \) equivalence classes then there is a DFA \( M \) accepting \( L \) with exactly \( n \) states and this is the minimum possible.

**Corollary**

A language \( L \) is non-regular if and only if there is an infinite fooling set \( F \) for \( L \).

**Algorithmic implication:** For every DFA \( M \) one can find in polynomial time a DFA \( M' \) such that \( L(M) = L(M') \) and \( M' \) has the fewest possible states among all such DFAs.
What was that all about

Summary: A regular language $L$ has a unique (up to naming) minimal automata, and it can be computed efficiently once any DFA is given for $L$. 
Exercise

1. Given two DFAs $M_1, M_2$ describe an efficient algorithm to decide if $L(M_1) = L(M_2)$.

2. Given DFA $M$, and two states $q, q'$ of $M$, show an efficient algorithm to decide if $q$ and $q'$ are distinguishable. (Hint: Use the first part.)

3. Given a DFA $M$ for a language $L$, describe an efficient algorithm for computing the minimal automata (as far as the number of states) that accepts $L$. 

Har-Peled (UIUC) CS374 Fall 2020