NFAs continued, Closure Properties of Regular Languages

Lecture 5
Tuesday, September 8, 2020
5.1 Equivalence of NFAs and DFAs
Regular Languages, DFAs, NFAs

Theorem

Languages accepted by DFAs, NFAs, and regular expressions are the same.

- DFAs are special cases of NFAs (easy)
- NFAs accept regular expressions (seen)
- DFAs accept languages accepted by NFAs (shortly)
- Regular expressions for languages accepted by DFAs (later in the course)
Regular Languages, DFAs, NFAs

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Equivalence of NFAs and DFAs

Theorem

*For every NFA $N$ there is a DFA $M$ such that $L(M) = L(N)$.***
5.1.1 The idea of the conversion of NFA to DFA
DFA\text{\textsf{s}}\ are\ memoryless…

1. DFA knows only its current state.
2. The state is the memory.
3. To design a DFA, answer the question:
   What minimal info needed to solve problem.
Simulating NFA

Example the first revisited

Previous lecture.. Ran NFA\(^{(N1)}\) on input \textit{ababa}.

$t = 0$:

$t = 1$:

$t = 2$:

$t = 3$:

$t = 4$:

$t = 5$:
The state of the NFA

It is easy to state that the state of the automata is the states that it might be situated at.

configuration: A set of states the automata might be in. Possible configurations: $\emptyset, \{A\}, \{A, B\}...$

Big idea: Build a DFA on the configurations.
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Possible configurations: $\emptyset$, \{A\}, \{A, B\}...

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Configuration: A set of states the automata might be in.
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configuration: A set of states the automata might be in.
Possible configurations: $\emptyset$, $\{A\}$, $\{A, B\}$...
Big idea: Build a **DFA** on the configurations.
Example

The formal construction based on the above idea is as follows. Consider an NFA \( N = (Q, \Delta, \overline{\delta}, s, A) \).

Define the DFA \( \text{det}(N) = (Q_0, \Delta, s_0, A_0) \) as follows.

- \( Q_0 = \mathcal{P}(Q) \)
- \( s_0 = \overline{\delta}(s, \epsilon) \)
- \( A_0 = \{ X \in Q_0 | X \notin A \} \)
- \( 0(X, a) = \overline{\delta}(X, a) \)

An example NFA is shown in Figure 4 along with the DFA \( \text{det}(N) \) in Figure 5.

We will now prove that the DFA defined above is correct. That is

**Lemma 4.** \( L(N) = L(\text{det}(N)) \)

**Proof.** Need to show \( \forall w \in \Delta^* \overline{\delta}(s_0, w) \in A_0 \iff \overline{\delta}(s, w) \in A \).

For the induction proof to go through we need to strengthen the claim as follows.

\[ \forall w \in \Delta^* \overline{\delta}(s_0, w) = \overline{\delta}(s, w) \]

In other words, this says that the state of the DFA after reading some string is exactly the set of states the NFA could be in after reading the same string.
Simulating an NFA by a DFA

- Think of a program with fixed memory that needs to simulate NFA $N$ on input $w$.
- What does it need to store after seeing a prefix $x$ of $w$?
  - It needs to know at least $\delta^*(s, x)$, the set of states that $N$ could be in after reading $x$.
  - Is it sufficient? Yes, if it can compute $\delta^*(s, xa)$ after seeing another symbol $a$ in the input.
- When should the program accept a string $w$? If $\delta^*(s, w) \cap A \neq \emptyset$.

Key Observation: DFA $M$ simulating $N$ should know current configuration of $N$.

State space of the DFA is $\mathcal{P}(Q)$. 
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**Key Observation:** DFA $M$ simulating $N$ should know current configuration of $N$.

State space of the DFA is $\mathcal{P}(Q)$. 
Example: DFA from NFA

NFA:

DFA:
Formal Tuple Notation for NFA

Definition

A non-deterministic finite automata (NFA) $N = (Q, \Sigma, \delta, s, A)$ is a five tuple where

- $Q$ is a finite set whose elements are called states,
- $\Sigma$ is a finite set called the input alphabet,
- $\delta : Q \times \Sigma \cup \{\epsilon\} \rightarrow \mathcal{P}(Q)$ is the transition function (here $\mathcal{P}(Q)$ is the power set of $Q$),
- $s \in Q$ is the start state,
- $A \subseteq Q$ is the set of accepting/final states.

$\delta(q, a)$ for $a \in \Sigma \cup \{\epsilon\}$ is a subset of $Q$ — a set of states.
THE END

...  

(for now)
5.1.2
Algorithm for converting NFA to DFA
Recall I
Extending the transition function to strings

**Definition**
For NFA \( N = (Q, \Sigma, \delta, s, A) \) and \( q \in Q \) the \( \epsilon \text{reach}(q) \) is the set of all states that \( q \) can reach using only \( \epsilon \)-transitions.

**Definition**
Inductive definition of \( \delta^* : Q \times \Sigma^* \rightarrow \mathcal{P}(Q) \):
- if \( w = \epsilon \), \( \delta^*(q, w) = \epsilon \text{reach}(q) \)
- if \( w = a \) where \( a \in \Sigma \): \( \delta^*(q, a) = \epsilon \text{reach}\left( \bigcup_{p \in \epsilon \text{reach}(q)} \delta(p, a) \right) \)
- if \( w = ax \): \( \delta^*(q, w) = \epsilon \text{reach}\left( \bigcup_{p \in \epsilon \text{reach}(q)} \bigcup_{r \in \delta^*(p, a)} \delta^*(r, x) \right) \)
Definition

A string $w$ is accepted by NFA $N$ if $\delta^*_N(s, w) \cap A \neq \emptyset$.

Definition

The language $L(N)$ accepted by a NFA $N = (Q, \Sigma, \delta, s, A)$ is

$$\{ w \in \Sigma^* \mid \delta^*(s, w) \cap A \neq \emptyset \}.$$
Subset Construction

NFA $N = (Q, \Sigma, s, \delta, A)$. We create a DFA $D = (Q', \Sigma, \delta', s', A')$ as follows:

- $Q' = \mathcal{P}(Q)$
- $s' = \varepsilon{\text{reach}}(s) = \delta^*(s, \epsilon)$
- $A' = \{X \subseteq Q \mid X \cap A \neq \emptyset\}$
- $\delta'(X, a) = \bigcup_{q \in X} \delta^*(q, a)$ for each $X \subseteq Q$, $a \in \Sigma$. 


Subset Construction

NFA \( \mathcal{N} = (Q, \Sigma, s, \delta, A) \). We create a DFA \( \mathcal{D} = (Q', \Sigma, \delta', s', A') \) as follows:

- \( Q' = \mathcal{P}(Q) \)
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- \( A' = \{X \subseteq Q \mid X \cap A \neq \emptyset\} \)
- \( \delta'(X, a) = \bigcup_{q \in X} \delta^*(q, a) \) for each \( X \subseteq Q, a \in \Sigma \).
Subset Construction

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Incremental construction

Only build states reachable from $s' = \epsilon \text{reach}(s)$ the start state of $D$

$\delta'(X, a) = \bigcup_{q \in X} \delta^*(q, a)$.
An optimization: Incremental algorithm

- Build $D$ beginning with start state $s' == \varepsilon \text{reach}(s)$
- For each existing state $X \subseteq Q$ consider each $a \in \Sigma$ and calculate the state $U = \delta'(X, a) = \bigcup_{q \in X} \delta^*(q, a)$ and add a transition.

To compute $Z_{q,a} = \delta^*(q, a)$ - set of all states reached from $q$ on character $a$
  - Compute $X_1 = \varepsilon \text{reach}(q)$
  - Compute $Y_1 = \bigcup_{p \in X_1} \delta(p, a)$
  - Compute $Z_{q,a} = \varepsilon \text{reach}(Y) = \bigcup_{r \in Y_1} \varepsilon \text{reach}(r)$
- If $U$ is a new state add it to reachable states that need to be explored.
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- If $U$ is a new state add it to reachable states that need to be explored.
THE END

... 

(for now)
5.1.3

Proof of correctness of conversion of NFA to DFA
Proof of Correctness

**Theorem**

Let $N = (Q, \Sigma, s, \delta, A)$ be a NFA and let $D = (Q', \Sigma, \delta', s', A')$ be a DFA constructed from $N$ via the subset construction. Then $L(N) = L(D)$. 

Stronger claim:

**Lemma**

For every string $w$, $\delta^*_N(s, w) = \delta^*_D(s', w)$.

Proof by induction on $|w|$. 

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**Stronger claim:**

**Lemma**

For every string \( w \), \( \delta_N^*(s, w) = \delta_D^*(s', w) \).

**Proof by induction on \( |w| \).**
Lemma

For every string $w$, $\delta^*_N(s, w) = \delta^*_D(s', w)$.

Proof:
Base case: $w = \epsilon$.

$\delta^*_N(s, \epsilon) = \epsilon\text{reach}(s)$.

$\delta^*_D(s', \epsilon) = s' = \epsilon\text{reach}(s)$ by definition of $s'$. 
Proof continued II

Lemma

For every string $w$, $\delta^*_N(s, w) = \delta^*_D(s', w)$.

Inductive step: $w = xa$ (Note: suffix definition of strings)

$\delta^*_N(s, xa) = \bigcup_{p \in \delta^*_N(s, x)} \delta^*_N(p, a)$ by inductive definition of $\delta^*_N$

$\delta^*_D(s', xa) = \delta_D(\delta^*_D(s, x), a)$ by inductive definition of $\delta^*_D$

By inductive hypothesis: $Y = \delta^*_N(s, x) = \delta^*_D(s, x)$

Thus $\delta^*_N(s, xa) = \bigcup_{p \in Y} \delta^*_N(p, a) = \delta_D(Y, a)$ by definition of $\delta_D$.

Therefore,

$\delta^*_N(s, xa) = \delta_D(Y, a) = \delta_D(\delta^*_D(s, x), a) = \delta^*_M(s', xa)$. which is what we need.
Lemma

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THE END
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(for now)
5.2 Closure Properties of Regular Languages
Regular Languages

Regular languages have three different characterizations
- Inductive definition via base cases and closure under union, concatenation and Kleene star
- Languages accepted by DFAs
- Languages accepted by NFAs

Regular language closed under many operations:
- union, concatenation, Kleene star via inductive definition or NFAs
- complement, union, intersection via DFAs
- homomorphism, inverse homomorphism, reverse, ...

Different representations allow for flexibility in proofs.
Regular Languages

Regular languages have three different characterizations

- Inductive definition via base cases and closure under union, concatenation and Kleene star
- Languages accepted by **DFA**s
- Languages accepted by **NFA**s

Regular language closed under many operations:

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Different representations allow for flexibility in proofs.
Example: PREFIX

Let $L$ be a language over $\Sigma$.

**Definition**

$\text{PREFIX}(L) = \{w \mid wx \in L, x \in \Sigma^*\}$

**Theorem**

If $L$ is regular then $\text{PREFIX}(L)$ is regular.

Let $M = (Q, \Sigma, \delta, s, A)$ be a DFA that recognizes $L$

$X = \{q \in Q \mid s \text{ can reach } q \text{ in } M\}$

$Y = \{q \in Q \mid q \text{ can reach some state in } A\}$

$Z = X \cap Y$

Create new DFA $M' = (Q, \Sigma, \delta, s, Z)$

**Claim:** $L(M') = \text{PREFIX}(L)$. 
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Claim: $L(M') = \text{PREFIX}(L)$. 
Example: \textsc{PREFIX}

Let $L$ be a language over $\Sigma$.

\textbf{Definition}

$\textsc{PREFIX}(L) = \{w \mid wx \in L, x \in \Sigma^*\}$

\textbf{Theorem}

\begin{itemize}
\item \textit{If $L$ is regular then $\textsc{PREFIX}(L)$ is regular.}
\end{itemize}

Let $M = (Q, \Sigma, \delta, s, A)$ be a \textbf{DFA} that recognizes $L$

$X = \{q \in Q \mid s \text{ can reach } q \text{ in } M\}$

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Create new \textbf{DFA} $M' = (Q, \Sigma, \delta, s, Z)$

\textbf{Claim:} $L(M') = \textsc{PREFIX}(L)$.  

\textbf{Example: \textsc{PREFIX}}

Let $L$ be a language over $\Sigma$.

\begin{itemize}
\item \textbf{Definition}
\end{itemize}

$\textsc{PREFIX}(L) = \{w \mid wx \in L, x \in \Sigma^*\}$

\begin{itemize}
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\end{itemize}

\begin{itemize}
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Claim: $L(M') = \textsc{PREFIX}(L)$.  

**Example: PREFIX**

Let \( L \) be a language over \( \Sigma \).

**Definition**

\[
\text{PREFIX}(L) = \{ w \mid wx \in L, x \in \Sigma^* \}
\]

**Theorem**

If \( L \) is regular then \( \text{PREFIX}(L) \) is regular.

Let \( M = (Q, \Sigma, \delta, s, A) \) be a DFA that recognizes \( L \)

\[
X = \{ q \in Q \mid s \text{ can reach } q \text{ in } M \} \\
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Z = X \cap Y
\]

Create new DFA \( M' = (Q, \Sigma, \delta, s, Z) \)

**Claim:** \( L(M') = \text{PREFIX}(L) \).
Example: PREFIX

Let \( L \) be a language over \( \Sigma \).

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\[
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\[
Z = X \cap Y
\]

Create new DFA \( M' = (Q, \Sigma, \delta, s, Z) \)

**Claim:** \( L(M') = \text{PREFIX}(L) \).
Exercise: SUFFIX

Let $L$ be a language over $\Sigma$.

**Definition**

$\text{SUFFIX}(L) = \{w \mid xw \in L, x \in \Sigma^*\}$

Prove the following:

**Theorem**

*If $L$ is regular then $\text{PREFIX}(L)$ is regular.*
Exercise: SUFFIX

An alternative “proof” using a figure
THE END

... (for now)
5.3

Algorithm for converting **NFA** into regular expression
Stage 0: Input

A → B

B: a, b

A, C: a

C: b

A, B: b

A, C: a, b
Stage 1: Normalizing

\[
\begin{align*}
A &\xrightarrow{a} B \\
&\xleftarrow{b} C \\
&\xrightarrow{a, b} C \\
C &\xrightarrow{a, b} C
\end{align*}
\]

\[
\begin{align*}
\Rightarrow
\end{align*}
\]

\[
\begin{align*}
\text{init} &\xrightarrow{\epsilon} A \\
A &\xrightarrow{a} B \\
&\xrightarrow{b} C \\
&\xrightarrow{a + b} AC
\end{align*}
\]
Stage 2: Remove state A
Stage 4: Redrawn without old edges
Stage 4: Removing B

\[ \text{init} \xrightarrow{a} B \xrightarrow{b} \text{C} \xrightarrow{\varepsilon} \text{AC} \]

\[ \text{init} \xrightarrow{a} B \xrightarrow{b} \text{C} \xrightarrow{\varepsilon} \text{AC} \]

\[ a + b \]

\[ a \]

\[ \text{ab} \ast \text{a} \]

\[ \text{init} \xrightarrow{b} \text{B} \xrightarrow{b} \text{b} \]

\[ a \]

\[ \varepsilon \]

\[ a + b \]}
Stage 5: Redraw

init → B
init → C
init → AC

C → init
C → AC

ε → C
ε → AC

a + b
ab^*a
b
a

⇒

init → C
init → AC

ab^*a + b
A
ε

a + b
Stage 6: Removing C

\[ \begin{align*}
\text{init} & \rightarrow C \\
C & \xrightarrow{\epsilon} AC \\
a + b & \xrightarrow{\epsilon} AC
\end{align*} \]

\[ (ab^*a + b)(a + b)^* \epsilon \]

\[ \begin{align*}
\text{init} & \rightarrow C \\
C & \xrightarrow{\epsilon} AC \\
a + b & \xrightarrow{\epsilon} AC
\end{align*} \]
Stage 7: Redraw

\[(ab^* a + b)(a + b)^* \epsilon\]
Stage 8: Extract regular expression

Thus, this automata is equivalent to the regular expression

\[(ab^*a + b)(a + b)^*\].
THE END

...  

(for now)