Deterministic Finite Automata (DFAs)

Lecture 3
Tuesday, September 1, 2020
3.1 DFA Introduction
DFAs also called Finite State Machines (FSMs)

- The “simplest” model for computers?
- State machines that are common in practice.
  - Vending machines
  - Elevators
  - Digital watches
  - Simple network protocols
- Programs with fixed memory
A simple program

Program to check if a given input string $w$ has odd length

```
int n = 0
While input is not finished
    read next character c
    n ← n + 1
endWhile
If (n is odd) output YES
Else output NO
```

```
bit x = 0
While input is not finished
    read next character c
    x ← flip(x)
endWhile
If (x = 1) output YES
Else output NO
```
A simple program

Program to check if a given input string $w$ has odd length

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If (n is odd) output YES
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    read next character c
    x ← flip(x)
endWhile
If (x = 1) output YES
Else output NO
```
Another view

- Machine has input written on a read-only tape
- Start in specified start state
- Start at left, scan symbol, change state and move right
- Circled states are accepting
- Machine accepts input string if it is in an accepting state after scanning the last symbol.
Draw me a DFA

DFA to check if a given input string has odd length
THE END

...(for now)
3.1.1

Graphical representation of DFA
Directed graph with nodes representing **states** and edge/arcs representing **transitions** labeled by symbols in $\Sigma$.

For each state (vertex) $q$ and symbol $a \in \Sigma$ there is **exactly** one outgoing edge labeled by $a$.

Initial/start state has a pointer (or labeled as $s$, $q_0$ or “start”).

Some states with double circles labeled as accepting/final states.
Where does 001 lead?
Where does 10010 lead?
Which strings end up in accepting state?
Can you prove it?
Every string \( w \) has a unique walk that it follows from a given state \( q \) by reading one letter of \( w \) from left to right.
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Which strings end up in accepting state?
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Which strings end up in accepting state?
Can you prove it?
Every string $w$ has a unique walk that it follows from a given state $q$ by reading one letter of $w$ from left to right.
**Definition**

A DFA $M$ accepts a string $w$ iff the unique walk starting at the start state and spelling out $w$ ends in an accepting state.

**Definition**

The language accepted (recognized) by a DFA $M$ is denoted by $L(M)$ and is formally defined as:

$L(M) = \{ w | M \text{ accepts } w \}$.
Definition

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The language accepted (or recognized) by a DFA $M$ is denoted by $L(M)$ and defined as:

$$L(M) = \{ w \mid M \text{ accepts } w \}.$$
“$M$ accepts language $L$” does not mean simply that that $M$ accepts each string in $L$.

It means that $M$ accepts each string in $L$ and no others. Equivalently $M$ accepts each string in $L$ and does not accept/rejects strings in $\Sigma^* \setminus L$.

$M$ “recognizes” $L$ is a better term but “accepts” is widely accepted (and recognized) (joke attributed to Lenny Pitt)
Warning

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THE END

... 

(for now)
3.1.2

Formal definition of DFA
**Definition**

A deterministic finite automata (DFA) $M = (Q, \Sigma, \delta, s, A)$ is a five tuple where

- $Q$ is a finite set whose elements are called states,
- $\Sigma$ is a finite set called the input alphabet,
- $\delta : Q \times \Sigma \rightarrow Q$ is the transition function,
- $s \in Q$ is the start state,
- $A \subseteq Q$ is the set of accepting/final states.

Common alternate notation: $q_0$ for start state, $F$ for final states.
Formal Tuple Notation

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Common alternate notation: $q_0$ for start state, $F$ for final states.
DFA Notation

\[ M = \left( \hat{Q}, \sum, \delta, s, \hat{A} \right) \]

- \( \hat{Q} \): set of all states
- \( \sum \): alphabet
- \( \delta \): transition function
- \( s \): start state
- \( \hat{A} \): set of all accept states
\( Q = \{ q_0, q_1, q_1, q_3 \} \)
\( \Sigma = \{ 0, 1 \} \)
\( \delta \)
\( s = q_0 \)
\( A = \{ q_0 \} \)
Example

- \( Q = \{ q_0, q_1, q_1, q_3 \} \)
- \( \Sigma = \{ 0, 1 \} \)
- \( \delta \)
- \( s = q_0 \)
- \( A = \{ q_0 \} \)
Example

- $Q = \{q_0, q_1, q_1, q_3\}$
- $\Sigma = \{0, 1\}$
- $\delta$
- $s = q_0$
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- $\Sigma = \{0, 1\}$
- $\delta$
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Example: The transition function

\[
\begin{array}{c|c|c}
\text{state} & \text{input} & \text{result} \\
\hline
q & c & \delta(q, c) \\
\hline
q_0 & 0 & q_3 \\
q_0 & 1 & q_1 \\
q_1 & 0 & q_0 \\
q_1 & 1 & q_2 \\
q_2 & 0 & q_2 \\
q_2 & 1 & q_2 \\
q_3 & 0 & q_2 \\
q_3 & 1 & q_0 \\
\end{array}
\]
THE END

... 

(for now)
3.1.3

Extending the transition function to strings
Extending the transition function to strings

Given DFA $M = (Q, \Sigma, \delta, s, A)$, $\delta(q, a)$ is the state that $M$ goes to from $q$ on reading letter $a$.

Useful to have notation to specify the unique state that $M$ will reach from $q$ on reading string $w$.

Transition function $\delta^* : Q \times \Sigma^* \rightarrow Q$ defined inductively as follows:

- $\delta^*(q, \epsilon) = q$ if $w = \epsilon$
- $\delta^*(q, ax) = \delta^*(\delta(q, a), x)$ if $w = ax$. 
Extending the transition function to strings

Given DFA $M = (Q, \Sigma, \delta, s, A)$, $\delta(q, a)$ is the state that $M$ goes to from $q$ on reading letter $a$

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Transition function $\delta^* : Q \times \Sigma^* \rightarrow Q$ defined inductively as follows:

- $\delta^*(q, w) = q$ if $w = \epsilon$
- $\delta^*(q, w) = \delta^*(\delta(q, a), x)$ if $w = ax$. 
Formal definition of language accepted by $M$

Definition

The language $L(M)$ accepted by a DFA $M = (Q, \Sigma, \delta, s, A)$ is

$$\{ w \in \Sigma^* \mid \delta^*(s, w) \in A \}.$$
What is:

- $\delta^*(q_1, \epsilon)$
- $\delta^*(q_0, 1011)$
- $\delta^*(q_1, 010)$
- $\delta^*(q_4, 10)$
- So what is $L(M)$???????
What is:

- $\delta^*(q_1, \epsilon)$
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What is:

\[ \delta^*(q_1, \epsilon) \]
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So what is \( L(M) \)
What is:

- $\delta^*(q_1, \epsilon)$
- $\delta^*(q_0, 1011)$
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So what is $L(M)$?
What is:

- $\delta^*(q_1, \epsilon)$
- $\delta^*(q_0, 1011)$
- $\delta^*(q_1, 010)$
- $\delta^*(q_4, 10)$
- So what is $L(M)$??????
What is $L(M)$ if start state is changed to $q_1$?
What is $L(M)$ if final/accept states are set to \{q_2, q_3\} instead of \{q_0\}?
Example continued

What is \( L(M) \) if final/accept states are set to \( \{q_2, q_3\} \) instead of \( \{q_0\} \)?
What is $L(M)$ if final/accept states are set to $\{q_2, q_3\}$ instead of $\{q_0\}$?
Advantages of formal specification

- Necessary for proofs
- Necessary to specify abstractly for class of languages

**Exercise:** Prove by induction that for any two strings $u, v$, any state $q$, $\delta^*(q, uv) = \delta^*(\delta^*(q, u), v)$. 
THE END

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(for now)
3.2

Constructing DFAs
How do we design a DFA $M$ for a given language $L$? That is $L(M) = L$.

- **DFA** is a like a program that has fixed amount of memory independent of input size.
- The memory of a DFA is encoded in its states.
- The state/memory must capture enough information from the input seen so far that it is sufficient for the suffix that is yet to be seen (note that DFA cannot go back).
DFA Construction: Examples

Example I: Basic languages

Assume $\Sigma = \{0, 1\}$.

$L = \emptyset$, $L = \Sigma^*$, $L = \{\epsilon\}$, $L = \{0\}$.  

DFA Construction: Examples

Example II: Length divisible by 5

Assume $\Sigma = \{0, 1\}$.

$L = \{w \in \{0, 1\}^* \mid |w| \text{ is divisible by 5}\}$
DFA Construction: examples

Example III: Ends with 01

Assume $\Sigma = \{0, 1\}$.

$L = \{w \in \{0, 1\}^* \mid w \text{ ends with } 01\}$
DFA Construction: examples

Example IV: Contains 001

Assume $\Sigma = \{0, 1\}$.

$L = \{w \in \{0, 1\}^* \mid w \text{ contains 001 as substring}\}$
DFA Construction: examples

Example V: Contains 001 or 010

Assume $\Sigma = \{0, 1\}$.

$L = \{w \in \{0, 1\}^* \mid w \text{ contains 001 or 010 as substring}\}$
Assume $\Sigma = \{0, 1\}$.

$L = \{w \mid w \text{ has a } 1 \text{ } k \text{ positions from the end}\}$. 

**DFA Construction: Example**

\( L = \{ \text{Binary numbers congruent to } 0 \mod 5 \} \)

Example:

1. \(110101_2 = 107_{10} = 2 \mod 5,\)
2. \(1010_2 = 10 = 0 \mod 5\)

**Key observation:**

\(\text{val}(w) \mod 5 = a\) implies

\[\begin{align*}
\text{val}(w_0) \mod 5 &= (\text{val}(w) \times 2) \mod 5 = 2a \mod 5 \\
\text{val}(w_1) \mod 5 &= (\text{val}(w) \times 2 + 1) \mod 5 = (2a + 1) \mod 5
\end{align*}\]
DFA Construction: Example

$L = \{\text{Binary numbers congruent to 0 mod 5}\}$

Example:

1. $1101011_2 = 107_{10} = 2 \mod 5$
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Key observation:

$val(w) \mod 5 = a$ implies

\[
val(w0) \mod 5 = (val(w) \cdot 2) \mod 5 = 2a \mod 5
\]

\[
val(w1) \mod 5 = (val(w) \cdot 2 + 1) \mod 5 = (2a + 1) \mod 5
\]
THE END

...  

(for now)
3.3
Complement language
Complement

**Question:** If \( M \) is a DFA, is there a DFA \( M' \) such that \( L(M') = \Sigma^* \setminus L(M) \)? That is, are languages recognized by DFAs closed under complement?
Complement

Example...

Just flip the state of the states!
**Theorem**

Languages accepted by **DFA**s are closed under complement.

**Proof.**

Let $M = (Q, \Sigma, \delta, s, A)$ such that $L = L(M)$. Let $M' = (Q, \Sigma, \delta, s, Q \setminus A)$. Claim: $L(M') = \overline{L}$. Why?

$\delta^*_M = \delta^*_M'$. Thus, for every string $w$, $\delta^*_M(s, w) = \delta^*_M'(s, w)$.

$\delta^*_M(s, w) \in A \Rightarrow \delta^*_M'(s, w) \notin Q \setminus A$. $\delta^*_M(s, w) \notin A \Rightarrow \delta^*_M'(s, w) \in Q \setminus A$. $\square$
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Languages accepted by DFA's are closed under complement.

**Proof.**

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$\square$
THE END

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(for now)
3.4
Product Construction
Union and Intersection

**Question:** Are languages accepted by **DFA**s closed under union? That is, given **DFA**s \(M_1\) and \(M_2\) is there a **DFA** that accepts \(L(M_1) \cup L(M_2)\)? How about intersection \(L(M_1) \cap L(M_2)\)?

Idea from programming: on input string \(w\)
- Simulate \(M_1\) on \(w\)
- Simulate \(M_2\) on \(w\)
- If both accept then \(w \in L(M_1) \cap L(M_2)\). If at least one accepts then \(w \in L(M_1) \cup L(M_2)\).
- **Catch:** We want a single **DFA** \(M\) that can only read \(w\) once.
- **Solution:** Simulate \(M_1\) and \(M_2\) in parallel by keeping track of states of both machines.
Question: Are languages accepted by DFAs closed under union? That is, given DFAs $M_1$ and $M_2$ is there a DFA that accepts $L(M_1) \cup L(M_2)$? How about intersection $L(M_1) \cap L(M_2)$?

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- **Catch:** We want a single DFA $M$ that can only read $w$ once.
- **Solution:** Simulate $M_1$ and $M_2$ in parallel by keeping track of states of both machines
Example

\[ M_1 \text{ accepts } #0 = \text{ odd} \]

\[ M_2 \text{ accepts } #1 = \text{ odd} \]
Example

$M_1$ accepts $\#0 = \text{odd}$

$M_2$ accepts $\#1 = \text{odd}$

*Cross-product machine*
Example II

Accept all binary strings of length divisible by 3 and 5
Product construction for intersection

\[ M_1 = (Q_1, \Sigma, \delta_1, s_1, A_1) \] and \[ M_2 = (Q_1, \Sigma, \delta_2, s_2, A_2) \]

Create \[ M = (Q, \Sigma, \delta, s, A) \] where

- \[ Q = Q_1 \times Q_2 = \{(q_1, q_2) \mid q_1 \in Q_1, q_2 \in Q_2\} \]
- \[ s = (s_1, s_2) \]
- \[ \delta : Q \times \Sigma \rightarrow Q \] where
  \[ \delta((q_1, q_2), a) = (\delta_1(q_1, a), \delta_2(q_2, a)) \]
- \[ A = A_1 \times A_2 = \{(q_1, q_2) \mid q_1 \in A_1, q_2 \in A_2\} \]

Theorem

\[ L(M) = L(M_1) \cap L(M_2). \]
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- \( s = (s_1, s_2) \)
- \( \delta : Q \times \Sigma \to Q \) where
  \[
  \delta((q_1, q_2), a) = (\delta_1(q_1, a), \delta_2(q_2, a))
  \]
- \( A = A_1 \times A_2 = \{(q_1, q_2) \mid q_1 \in A_1, q_2 \in A_2\} \)

**Theorem**

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- \( s = (s_1, s_2) \)
- \( \delta : Q \times \Sigma \to Q \) where

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\delta((q_1, q_2), a) = (\delta_1(q_1, a), \delta_2(q_2, a))
\]

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Theorem

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Theorem

\[ L(M) = L(M_1) \cap L(M_2). \]
Product construction for intersection

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Correctness of construction

**Lemma**

For each string $w$, $\delta^*(s, w) = (\delta_1^*(s_1, w), \delta_2^*(s_2, w))$.

**Exercise:** Assuming lemma prove the theorem in previous slide. Proof of lemma by induction on $|w|$. 
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Product construction for union

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\[ L(M) = L(M_1) \cup L(M_2). \]
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**Theorem**

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Set Difference

**Theorem**

$M_1, M_2$ DFAs. There is a DFA $M$ such that $L(M) = L(M_1) \setminus L(M_2)$.

**Exercise:** Prove the above using two methods.
- Using a direct product construction
- Using closure under complement and intersection and union
Question: Why are DFAs required to only move right? Can we allow DFA to scan back and forth? Caveat: Tape is read-only so only memory is in machine’s state.

- Can define a formal notion of a “2-way” DFA
- Can show that any language recognized by a 2-way DFA can be recognized by a regular (1-way) DFA
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THE END

...(for now)
3.5
Supplemental: DFA philosophy
A finite program can be simulated by a DFA...

1. Finite program = a program that uses a prespecified bounded amount of memory.
2. Given DFA and input, easy to decide if DFA accepts input.
3. A finite program is a DFA!
   \# of states of memory of a finite program = finite.
   \# states \approx 2^{\# of memory bits used by program}
4. Program using 1K memory = has...
5. Turing halting theorem: Not possible (in general) to decide if a program stops on an input.
6. DFA \neq programs.
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But universe is finite...

1. Estimate \# of atoms in the universe is $10^{82}$.
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4. So... All programs in this universe are DFA's.
5. Checkmate Mate!
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So what is going on...

1. Theory models the world. (Oversimplifies it.)
2. Make it possible to think about it.
3. There are cases where theory does not model the world well.
4. Know when to apply the theory.
5. Reject statements that are correct but not useful.
6. Really Large finite numbers are
THE END

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