24.4.2
The consistency of execution
The variables of $\varphi$

**Variables:**

$\langle q_j, b_j, q'_j, b'_j, d_j \rangle$: $j$th instruction of $M$

$I(j, i)$: Instruction $j$ was issued at time $i$.

$H(h, i)$: The head is at location $h$ at time $i$.

$T(c, h, i)$: The tape at location $h$ at time $i$ stored the character $c$. 

$\varphi_1$: The input is encoded correctly

$\varphi_1$ asserts (is true iff) the variables are set T/F indicating that $M$ starts in state $q_0$ at time 0 with tape contents containing $x$ followed by blanks. Let $x = x_1x_2 \ldots x_n$

$$\varphi_1 = S(q_0, 0)$$

$$\bigwedge_{h=1}^{n} T(x_h, h, 0)$$  // at time 0 cells 1 to $n$ have value $x_1$ to $x_n$

$$\bigwedge_{h=n+1}^{p(n)} T(\bot, h, 0)$$  // all remaining cells are blank

$$\bigwedge H(1, 0)$$  // The head is at time 0 at start of tape
\( \varphi_2: M \) is in exactly one state at any point in time

\( \varphi_2 \) asserts \( M \) in exactly one state at any time \( i \):

\[
\varphi_2 = \bigwedge_{i=0}^{p(|x|)} \left( \bigoplus (S(q_0, i), S(q_1, i), \ldots, S(q_{|Q|}, i)) \right)
\]

**Variables:**

\( \langle q_j, b_j, q'_j, b'_j, d_j \rangle \): \( j \)th instruction of \( M \)

\( l(j, i) \): Instruction \( j \) was issued at time \( i \).

\( H(h, i) \): The head is at location \( h \) at time \( i \).

\( T(c, h, i) \): The tape at location \( h \) at time \( i \) stored the character \( c \).
\( \varphi_3: \) Each tape cell holds a unique symbol at any time

\( \varphi_3 \) asserts that each tape cell holds a unique symbol at any given time.

\[
\varphi_3 = \bigwedge_{i=0}^{p(|x|)} \bigwedge_{h=1}^{p(|x|)} \bigoplus (T(b_1, h, i), T(b_2, h, i), \ldots, T(b_{|\Gamma|}, h, i))
\]

For each time \( i \) and for each cell position \( h \) exactly one symbol \( b \in \Gamma \) at cell position \( h \) at time \( i \)

**Variables:**

- \( \langle q_j, b_j, q'_j, b'_j, d_j \rangle \): \( j \)th instruction of \( M \)
- \( I(j, i) \): Instruction \( j \) was issued at time \( i \).
- \( H(h, i) \): The head is at location \( h \) at time \( i \).
- \( T(c, h, i) \): The tape at location \( h \) at time \( i \) stored the character \( c \).
$\varphi_4$: tape head of $M$ is in exactly one position at any time $i$

$\varphi_4$ asserts that the read/write head of $M$ is in exactly one position at any time $i$

$$\varphi_4 = \bigwedge_{i=0}^{p(|x|)} (\oplus (H(1, i), H(2, i), \ldots, H(p(|x|), i)))$$

**Variables:**

$\langle q_j, b_j, q'_j, b'_j, d_j \rangle$: $j$th instruction of $M$

$I(j, i)$: Instruction $j$ was issued at time $i$.

$H(h, i)$: The head is at location $h$ at time $i$.

$T(c, h, i)$: The tape at location $h$ at time $i$ stored the character $c$. 
\( \varphi_5 \): \( M \) accepts the input

\( \varphi_5 \) asserts that \( M \) accepts

- Let \( q_a \) be unique accept state of \( M \)
- without loss of generality assume \( M \) runs all \( p(|x|) \) steps

\( \varphi_5 = S(q_a, p(|x|)) \)

State at time \( p(|x|) \) is \( q_a \) the accept state.

If we don’t want to make assumption of running for all steps

\[ \varphi_5 = \bigvee_{i=1}^{p(|x|)} S(q_a, i) \]

which means \( M \) enters accepts state at some time.
\( \phi_6: \) \( M \) executes a unique instruction at each time

\( \phi_6 \) asserts that \( M \) executes a unique instruction at each time

\[
\phi_6 = \prod_{i=0}^{p(|x|)} \bigoplus (l(1, i), l(2, i), \ldots, l(m, i))
\]

where \( m \) is max instruction number.

**Variables:**

\( \langle q_j, b_j, q'_j, b'_j, d_j \rangle \): \( j \)th instruction of \( M \)

\( l(j, i) \): Instruction \( j \) was issued at time \( i \).

\( H(h, i) \): The head is at location \( h \) at time \( i \).

\( T(c, h, i) \): The tape at location \( h \) at time \( i \) stored the character \( c \).
$\varphi_7$: Tape changes only because of the head writing something

$\varphi_7$ ensures that variables don’t allow tape to change from one moment to next if the read/write head was not there.

“If head is not at position $h$ at time $i$ then at time $i + 1$ the symbol at cell $h$ must be unchanged”

$$\varphi_7 = \bigwedge_i \bigwedge_h \bigwedge_{b \neq c} \left( H(h, i) \Rightarrow T(b, h, i) \land T(c, h, i + 1) \right)$$

since $A \Rightarrow B$ is same as $\neg A \lor B$, rewrite above in CNF form

$$\varphi_7 = \bigwedge_i \bigwedge_h \bigwedge_{b \neq c} \left( H(h, i) \lor \neg T(b, h, i) \lor \neg T(c, h, i + 1) \right)$$
\( \varphi_8: \) Transitions are done from correct states

\( j \)th instruction of \( M: < q_j, b_j, q'_j, b'_j, d_j > \)

\[
\varphi_8 = \bigwedge_i \bigwedge_j (I(j, i) \Rightarrow S(q_j, i))
\]

If instruction \( j \) is executed at time \( i \) then state at time \( i \) must be \( q_j \).

---

**Variables:**

\( < q_j, b_j, q'_j, b'_j, d_j > \): \( j \)th instruction of \( M \)

\( I(j, i) \): Instruction \( j \) was issued at time \( i \).

\( H(h, i) \): The head is at location \( h \) at time \( i \).

\( T(c, h, i) \): The tape at location \( h \) at time \( i \) stored the character \( c \).
$\varphi_9$: Transitions are done into correct state

$j$th instruction of $M$: $<q_j, b_j, q'_j, b'_j, d_j>$

$$\varphi_9 = \bigwedge_i \bigwedge_j (I(j, i) \Rightarrow S(q'_j, i + 1))$$

If instruction $j$ was performed at time $i$, then state at time $i + 1$ must be $q'_j$.

Variables:

$\langle q_j, b_j, q'_j, b'_j, d_j \rangle$: $j$th instruction of $M$

$I(j, i)$: Instruction $j$ was issued at time $i$.

$H(h, i)$: The head is at location $h$ at time $i$.

$T(c, h, i)$: The tape at location $h$ at time $i$ stored the character $c$.  

86 / 94
$\varphi_{10}$: The character written on tape that triggered an instruction, is the correct one

$$\varphi_{10} = \bigwedge_i \bigwedge_h \bigwedge_j [l(j, i) \land H(h, i) \Rightarrow T(b_j, h, i)]$$

If instruction $j$ was executed at time $i$ and head was at position $h$, then cell $h$ has the symbol needed to issue instruction $j$ is written under the head location on the tape.

Variables:

$\langle q_j, b_j, q'_j, b'_j, d_j \rangle$: $j$th instruction of $M$

$l(j, i)$: Instruction $j$ was issued at time $i$.

$H(h, i)$: The head is at location $h$ at time $i$.

$T(c, h, i)$: The tape at location $h$ at time $i$ stored the character $c$. 
$\varphi_{11}$: The correct symbol was written to the tape at time $i$

\[
\varphi_{11} = \bigwedge_i \bigwedge_j \bigwedge_h \left[ (I(j, i) \land H(h, i)) \Rightarrow T(b'_j, h, i + 1) \right]
\]

If instruction $j$ was executed time $i$ with head at $h$, then at next time step symbol $b'_j$ was written in position $h$

Variables:

$\langle q_j, b_j, q'_j, b'_j, d_j \rangle$: $j$th instruction of $M$

$I(j, i)$: Instruction $j$ was issued at time $i$.

$H(h, i)$: The head is at location $h$ at time $i$.

$T(c, h, i)$: The tape at location $h$ at time $i$ stored the character $c$. 
\( \varphi_{12} : \) Head was moved in the right direction at time \( i \)

\[
\varphi_{12} = \bigwedge_i \bigwedge_j \bigwedge_h [(I(j, i) \land H(h, i)) \Rightarrow H(h + d_j, i + 1)]
\]

The head is moved properly according to instr \( j \).

**Variables:**

- \( \langle q_j, b_j, q'_j, b'_j, d_j \rangle \): \( j \)th instruction of \( M \)
- \( I(j, i) \): Instruction \( j \) was issued at time \( i \).
- \( H(h, i) \): The head is at location \( h \) at time \( i \).
- \( T(c, h, i) \): The tape at location \( h \) at time \( i \) stored the character \( c \).
THE END

...

(for now)