24.3.3

Showing NP-Completeness of 3-COLORING
24.3.3.1
The variable assignment gadget
3-Coloring is **NP-Complete**

- **3-Coloring** is in **NP**.
  - **Certificate:** for each node a color from \(\{1, 2, 3\}\).
  - **Certifier:** Check if for each edge \((u, v)\), the color of \(u\) is different from that of \(v\).
- **Hardness:** We will show \(3\text{-SAT} \leq_P 3\text{-Coloring}\).
Reduction idea

1. \( \varphi \): Given 3SAT formula (i.e., 3CNF formula).
2. \( \varphi \): variables \( x_1, \ldots, x_n \) and clauses \( C_1, \ldots, C_m \).
3. Create graph \( G_\varphi \) s.t. \( G_\varphi \) 3-colorable \( \iff \varphi \) satisfiable.
   - encode assignment \( x_1, \ldots, x_n \) in colors assigned nodes of \( G_\varphi \).
   - create triangle with node True, False, Base
   - for each variable \( x_i \) two nodes \( v_i \) and \( \bar{v}_i \) connected in a triangle with common Base
   - If graph is 3-colored, either \( v_i \) or \( \bar{v}_i \) gets the same color as True. Interpret this as a truth assignment to \( v_i \)
   - Need to add constraints to ensure clauses are satisfied (next phase)
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Assignment encoding using 3-coloring
THE END

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(for now)