Circuit satisfiability and Cook-Levin Theorem

Lecture 24
Thursday, December 3, 2020
24.1
Recap
Recap

**NP**: languages that have non-deterministic polynomial time algorithms

A language $L$ is *NP-Complete* if and only if

- $L$ is in **NP**
- for every $L'$ in **NP**, $L' \leq_p L$

$L$ is *NP-Hard* if for every $L'$ in **NP**, $L' \leq_p L$.

**Theorem 24.1 (Cook-Levin).**

$\text{SAT}$ is **NP-Complete**.
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**NP**: languages that have non-deterministic polynomial time algorithms

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*SAT* is NP-Complete.
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**NP**: languages that have non-deterministic polynomial time algorithms

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$L$ is **NP-Hard** if for every $L'$ in NP, $L' \leq_P L$.

**Theorem 24.1 (Cook-Levin).**

*SAT* is NP-Complete.
Pictorial View
Possible scenarios:

1. \( P = NP \).
2. \( P \neq NP \)

Question: Suppose \( P \neq NP \). Is every problem in \( NP \setminus P \) also \( NP\text{-Complete} \)?

Theorem 24.2 (Ladner).

If \( P \neq NP \) then there is a problem/language \( X \in NP \setminus P \) such that \( X \) is not \( NP\text{-Complete} \).
Possible scenarios:

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\textbf{Theorem 24.2 (Ladner).}

\textit{If} \( P \neq NP \) \textit{then there is a problem/language} \( X \in NP \setminus P \) \textit{such that} \( X \) \textit{is not NP-Complete.}
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**Theorem 24.2 (Ladner).**

*If \( P \neq NP \) then there is a problem/language \( X \in NP \setminus P \) such that \( X \) is not \( NP\text{-Complete} \).*
What do we know so far

1. Independent Set $\leq_p$ Clique, Clique $\leq_p$ Independent Set.
   $\implies$ Clique $\cong_p$ Independent Set.

2. Vertex Cover $\leq_p$ Independent Set, Independent Set $\leq_p$ Vertex Cover.
   $\implies$ Independent Set $\cong_p$ Vertex Cover.

3. 3SAT $\leq_p$ SAT, SAT $\leq_p$ 3SAT $\implies$ 3SAT $\cong_p$ SAT.

4. 3SAT $\leq_p$ Independent Set.
   Exercise (or Cook-Levin theorem): Independent Set $\leq_p$ SAT
   $\implies$ 3SAT $\cong_p$ Independent Set.

5. SAT $\leq_p$ Hamiltonian Cycle
   Exercise (or Cook-Levin theorem): Hamiltonian Cycle $\leq_p$ 3SAT
   $\implies$ Hamiltonian Cycle $\cong_p$ 3SAT

6. Clique $\cong_p$ Independent Set $\cong_p$ Vertex Cover $\cong_p$ 3SAT
   $\cong_p$ SAT $\cong_p$ Hamiltonian Cycle
What do we know so far

1. **Independent Set** $\leq_p$ **Clique**, **Clique** $\leq_p$ **Independent Set**.
   $\implies$ **Clique** $\cong_p$ **Independent Set**.

2. **Vertex Cover** $\leq_p$ **Independent Set**, **Independent Set** $\leq_p$ **Vertex Cover**.
   $\implies$ **Independent Set** $\cong_p$ **Vertex Cover**.

3. **3SAT** $\leq_p$ **SAT**, **SAT** $\leq_p$ **3SAT** $\implies$ **3SAT** $\cong_p$ **SAT**.

4. **3SAT** $\leq_p$ **Independent Set**.
   Exercise (or Cook-Levin theorem): **Independent Set** $\leq_p$ **SAT**
   $\implies$ **3SAT** $\cong_p$ **Independent Set**.

5. **SAT** $\leq_p$ **Hamiltonian Cycle**
   Exercise (or Cook-Levin theorem): **Hamiltonian Cycle** $\leq_p$ **3SAT**
   $\implies$ **Hamiltonian Cycle** $\cong_p$ **3SAT**

6. **Clique** $\cong_p$ **Independent Set** $\cong_p$ **Vertex Cover** $\cong_p$ **3SAT**
   $\cong_p$ **SAT** $\cong_p$ **Hamiltonian Cycle**
What do we know so far

1. **Independent Set** $\leq_P$ **Clique**, **Clique** $\leq_P$ **Independent Set**.
   $\Rightarrow$ **Clique** $\cong_P$ **Independent Set**.

2. **Vertex Cover** $\leq_P$ **Independent Set**, **Independent Set** $\leq_P$ **Vertex Cover**.
   $\Rightarrow$ **Independent Set** $\cong_P$ **Vertex Cover**.

3. **3SAT** $\leq_P$ **SAT**, **SAT** $\leq_P$ **3SAT** $\Rightarrow$ **3SAT** $\cong_P$ **SAT**.

4. **3SAT** $\leq_P$ **Independent Set**.
   Exercise (or Cook-Levin theorem): **Independent Set** $\leq_P$ **SAT**
   $\Rightarrow$ **3SAT** $\cong_P$ **Independent Set**.

5. **SAT** $\leq_P$ **Hamiltonian Cycle**
   Exercise (or Cook-Levin theorem): **Hamiltonian Cycle** $\leq_P$ **3SAT**
   $\Rightarrow$ **Hamiltonian Cycle** $\cong_P$ **3SAT**

6. **Clique** $\cong_P$ **Independent Set** $\cong_P$ **Vertex Cover** $\cong_P$ **3SAT**
   $\cong_P$ **SAT** $\cong_P$ **Hamiltonian Cycle**
What do we know so far

1. **Independent Set** \(\leq_p\) Clique, Clique \(\leq_p\) Independent Set.
   \(\implies\) Clique \(\cong_p\) Independent Set.

2. Vertex Cover \(\leq_p\) Independent Set, Independent Set \(\leq_p\) Vertex Cover.
   \(\implies\) Independent Set \(\cong_p\) Vertex Cover.

3. 3SAT \(\leq_p\) SAT, SAT \(\leq_p\) 3SAT \(\implies\) 3SAT \(\cong_p\) SAT.

4. 3SAT \(\leq_p\) Independent Set.
   Exercise (or Cook-Levin theorem): **Independent Set** \(\leq_p\) SAT
   \(\implies\) 3SAT \(\cong_p\) Independent Set.

5. SAT \(\leq_p\) Hamiltonian Cycle
   Exercise (or Cook-Levin theorem): **Hamiltonian Cycle** \(\leq_p\) 3SAT
   \(\implies\) Hamiltonian Cycle \(\cong_p\) 3SAT

6. Clique \(\cong_p\) Independent Set \(\cong_p\) Vertex Cover \(\cong_p\) 3SAT
   \(\cong_p\) SAT \(\cong_p\) Hamiltonian Cycle
What do we know so far

1. **Independent Set** \( \leq_p \) **Clique**, **Clique** \( \leq_p \) **Independent Set**.
   \[ \implies \text{Clique} \cong_p \text{Independent Set}. \]

2. **Vertex Cover** \( \leq_p \) **Independent Set**, **Independent Set** \( \leq_p \) **Vertex Cover**.
   \[ \implies \text{Independent Set} \cong_p \text{Vertex Cover}. \]

3. **3SAT** \( \leq_p \) **SAT**, **SAT** \( \leq_p \) **3SAT** \[ \implies \text{3SAT} \cong_p \text{SAT}. \]

4. **3SAT** \( \leq_p \) **Independent Set**
   
   Exercise (or Cook-Levin theorem): **Independent Set** \( \leq_p \) **SAT**
   \[ \implies \text{3SAT} \cong_p \text{Independent Set}. \]

5. **SAT** \( \leq_p \) **Hamiltonian Cycle**
   
   Exercise (or Cook-Levin theorem): **Hamiltonian Cycle** \( \leq_p \) **3SAT**
   \[ \implies \text{Hamiltonian Cycle} \cong_p \text{3SAT}. \]

6. **Clique** \( \cong_p \) **Independent Set** \( \cong_p \) **Vertex Cover** \( \cong_p \) **3SAT**
   \[ \cong_p \text{SAT} \cong_p \text{Hamiltonian Cycle}. \]
What do we know so far

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NP Completeness

Clique $\cong_p$ Independent Set $\cong_p$ Vertex Cover $\cong_p$ 3SAT $\cong_p$ SAT $\cong_p$ Hamiltonian Cycle

All these problems are in $\text{NP}$.

$\text{SAT}$ is $\text{NPC}$.

All these problems are $\text{NP-Complete}$. 
NP Completeness

Clique $\approx_p$ Independent Set $\approx_p$ Vertex Cover $\approx_p$ 3SAT $\approx_p$ SAT $\approx_p$ Hamiltonian Cycle

All these problems are in **NP**.

**SAT** is **NPC**.

All these problems are **NP-Complete**.
NP Completeness

Clique $\equiv_P$ Independent Set $\equiv_P$ Vertex Cover $\equiv_P$ 3SAT $\equiv_P$ SAT $\equiv_P$ Hamiltonian Cycle

All these problems are in \textbf{NP}.

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NP Completeness

Clique $\cong_p$ Independent Set $\cong_p$ Vertex Cover $\cong_p$ 3SAT $\cong_p$ SAT $\cong_p$ Hamiltonian Cycle

All these problems are in $\textbf{NP}$.

\textbf{SAT} is $\textbf{NPC}$.

All these problems are $\textbf{NP-Complete}$. 
THE END

...(for now)