23.3.4

If there is a Hamiltonian cycle $\Rightarrow \exists$ satisfying assignment
Reduction: Hamiltonian cycle $\iff \exists$ satisfying assignment

We are given a Hamiltonian cycle in $G_\varphi$:

$x_1 \lor \neg x_2 \lor x_4$

$\neg x_1 \lor \neg x_2 \lor \neg x_3$

Want to extract satisfying assignment...
Reduction: Hamiltonian cycle $\iff \exists$ satisfying assignment

No shenanigan: Hamiltonian cycle cannot leave a row in the middle
Reduction: Hamiltonian cycle $\iff \exists$ satisfying assignment

No shenanigan: Hamiltonian cycle cannot leave a row in the middle
Reduction: Hamiltonian cycle \[\iff\exists\text{ satisfying assignment}\]

No shenanigan: Hamiltonian cycle cannot leave a row in the middle
Reduction: Hamiltonian cycle $\iff \exists$ satisfying assignment

No shenanigan: Hamiltonian cycle can not leave a row in the middle

Conclude: Hamiltonian cycle must go through each row completely from left to right, or right to left. As such, can be interpreted as a valid assignment.
Hamiltonian Cycle $\Rightarrow$ Satisfying assignment

Suppose $\Pi$ is a Hamiltonian cycle in $G_\varphi$

- If $\Pi$ enters $c_j$ (vertex for clause $C_j$) from vertex $3j$ on path $i$ then it must leave the clause vertex on edge to $3j + 1$ on the same path $i$
  - If not, then only unvisited neighbor of $3j + 1$ on path $i$ is $3j + 2$
  - Thus, we don’t have two unvisited neighbors (one to enter from, and the other to leave) to have a Hamiltonian Cycle

- Similarly, if $\Pi$ enters $c_j$ from vertex $3j + 1$ on path $i$ then it must leave the clause vertex $c_j$ on edge to $3j$ on path $i$
Thus, vertices visited immediately before and after $C_i$ are connected by an edge.

We can remove $c_j$ from cycle, and get Hamiltonian cycle in $G - c_j$.

Consider Hamiltonian cycle in $G - \{c_1, \ldots c_m\}$; it traverses each path in only one direction, which determines the truth assignment.
Hamiltonian Cycle $\implies$ Satisfying assignment (contd)

- Thus, vertices visited immediately before and after $C_i$ are connected by an edge.
- We can remove $c_j$ from cycle, and get Hamiltonian cycle in $G - c_j$.
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Hamiltonian Cycle \(\implies\) Satisfying assignment (contd)

- Thus, vertices visited immediately before and after \(C_i\) are connected by an edge
- We can remove \(c_j\) from cycle, and get Hamiltonian cycle in \(G - c_j\)
- Consider Hamiltonian cycle in \(G - \{c_1, \ldots c_m\}\); it traverses each path in only one direction, which determines the truth assignment

\[
\neg x_1 \lor \neg x_2 \lor \neg x_3
\]

\[
\begin{align*}
x_1 &= 0 \\
x_2 &= 1 \\
x_3 &= 0 \\
x_4 &= 1
\end{align*}
\]
Thus, vertices visited immediately before and after $C_i$ are connected by an edge.

We can remove $c_j$ from cycle, and get Hamiltonian cycle in $G - c_j$.

Consider Hamiltonian cycle in $G - \{c_1, \ldots c_m\}$; it traverses each path in only one direction, which determines the truth assignment.
Correctness Proof

We just proved:

**Lemma 23.2.**

\[ G_\varphi \text{ has a Hamiltonian cycle } \iff \varphi \text{ has a satisfying assignment } \alpha. \]

**Lemma 23.3.**

\( \varphi \) has a satisfying assignment iff \( G_\varphi \) has a Hamiltonian cycle.

**Proof.**

Follows from Lemma 23.1 and Lemma 23.2.
Correctness Proof

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\varphi \text{ has a satisfying assignment iff } G_\varphi \text{ has a Hamiltonian cycle.}

**Proof.**

Follows from Lemma 23.1 and Lemma 23.2.
Summary

What we did:

1. Showed that Directed Hamiltonian Cycle is in NP.
2. Provided a polynomial time reduction from 3SAT to Directed Hamiltonian Cycle.
3. Proved that $\varphi$ satisfiable $\iff G_\varphi$ is Hamiltonian.

Theorem 23.4.
The problem Hamiltonian Cycle in directed graphs is NP-Complete.
Summary

What we did:

1. Showed that Directed Hamiltonian Cycle is in NP.
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Theorem 23.4.
The problem Hamiltonian Cycle in directed graphs is NP-Complete.
THE END

...

(for now)