23.3

NP-Completeness of Hamiltonian Cycle
23.3.1
Reduction from 3SAT to Hamiltonian Cycle: Basic idea
Directed Hamiltonian Cycle

**Input**  Given a directed graph $G = (V, E)$ with $n$ vertices

**Goal**  Does $G$ have a Hamiltonian cycle?

▶ A Hamiltonian cycle is a cycle in the graph that visits every vertex in $G$ exactly once

![Graph diagram]
Directed Hamiltonian Cycle

**Input** Given a directed graph \( G = (V, E) \) with \( n \) vertices

**Goal** Does \( G \) have a Hamiltonian cycle?

► A Hamiltonian cycle is a cycle in the graph that visits every vertex in \( G \) exactly once
Is the following graph Hamiltonian?

(A) Yes.
(B) No.
Directed Hamiltonian Cycle is **NP-Complete**

- Directed Hamiltonian Cycle is in **NP**: exercise
- **Hardness**: We will show $3SAT \leq_P Directed\ Hamiltonian\ Cycle$. 
Reduction construction
From 3SAT to Hamiltonian cycle in directed graph

1. To show reduction, we next describe an algorithm:
   ▶ Input: 3SAT formula $\varphi$
   ▶ Output: A graph $G_{\varphi}$.
   ▶ Running time is polynomial.
   ▶ Requirement: $\varphi$ is satisfiable $\iff$ $G_{\varphi}$ is Hamiltonian.

2. Given 3SAT formula $\varphi$ create a graph $G_{\varphi}$ such that
   ▶ $G_{\varphi}$ has a Hamiltonian cycle if and only if $\varphi$ is satisfiable
   ▶ $G_{\varphi}$ should be constructible from $\varphi$ by a polynomial time algorithm $\mathcal{A}$

3. Notation: $\varphi$ has $n$ variables $x_1, x_2, \ldots, x_n$ and $m$ clauses $C_1, C_2, \ldots, C_m$. 
Reduction construction
From 3SAT to Hamiltonian cycle in directed graph

1. To show reduction, we next describe an algorithm:
   ▶ Input: 3SAT formula $\varphi$
   ▶ Output: A graph $G_\varphi$.
   ▶ Running time is polynomial.
   ▶ Requirement: $\varphi$ is satisfiable $\iff G_\varphi$ is Hamiltonian.

2. Given 3SAT formula $\varphi$ create a graph $G_\varphi$ such that
   ▶ $G_\varphi$ has a Hamiltonian cycle if and only if $\varphi$ is satisfiable
   ▶ $G_\varphi$ should be constructible from $\varphi$ by a polynomial time algorithm $A$

3. Notation: $\varphi$ has $n$ variables $x_1, x_2, \ldots, x_n$ and $m$ clauses $C_1, C_2, \ldots, C_m$. 
Reduction construction
From 3SAT to Hamiltonian cycle in directed graph

1. To show reduction, we next describe an algorithm:
   ▶ Input: 3SAT formula \( \varphi \)
   ▶ Output: A graph \( G_\varphi \).
   ▶ Running time is polynomial.
   ▶ Requirement: \( \varphi \) is satisfiable \( \iff \) \( G_\varphi \) is Hamiltonian.

2. Given 3SAT formula \( \varphi \) create a graph \( G_\varphi \) such that
   ▶ \( G_\varphi \) has a Hamiltonian cycle if and only if \( \varphi \) is satisfiable
   ▶ \( G_\varphi \) should be constructible from \( \varphi \) by a polynomial time algorithm \( \mathcal{A} \)

3. Notation: \( \varphi \) has \( n \) variables \( x_1, x_2, \ldots, x_n \) and \( m \) clauses \( C_1, C_2, \ldots, C_m \).
Reduction construction
From 3SAT to Hamiltonian cycle in directed graph

1. To show reduction, we next describe an algorithm:
   ▶ Input: **3SAT** formula $\varphi$
   ▶ Output: A graph $G_{\varphi}$.
   ▶ Running time is polynomial.
   ▶ Requirement: $\varphi$ is satisfiable $\iff G_{\varphi}$ is Hamiltonian.

2. Given **3SAT** formula $\varphi$ create a graph $G_{\varphi}$ such that
   ▶ $G_{\varphi}$ has a Hamiltonian cycle if and only if $\varphi$ is satisfiable
   ▶ $G_{\varphi}$ should be constructible from $\varphi$ by a polynomial time algorithm $A$

3. Notation: $\varphi$ has $n$ variables $x_1, x_2, \ldots, x_n$ and $m$ clauses $C_1, C_2, \ldots, C_m$. 
Reduction construction
From 3SAT to Hamiltonian cycle in directed graph

1. To show reduction, we next describe an algorithm:
   ▶ Input: 3SAT formula $\varphi$
   ▶ Output: A graph $G_{\varphi}$.
   ▶ Running time is polynomial.
   ▶ Requirement: $\varphi$ is satisfiable $\iff G_{\varphi}$ is Hamiltonian.

2. Given 3SAT formula $\varphi$ create a graph $G_{\varphi}$ such that
   ▶ $G_{\varphi}$ has a Hamiltonian cycle if and only if $\varphi$ is satisfiable
   ▶ $G_{\varphi}$ should be constructible from $\varphi$ by a polynomial time algorithm $A$

3. Notation: $\varphi$ has $n$ variables $x_1, x_2, \ldots, x_n$ and $m$ clauses $C_1, C_2, \ldots, C_m$. 
Encoding assignments

Converting $\varphi$ to a graph

Given a formula with $n$ variables, we need a graph with $2^n$ different Hamiltonian paths, that can encode their assignments.
Encoding assignments

Converting $\varphi$ to a graph

Given a formula with $n$ variables, we need a graph with $2^n$ different Hamiltonian paths, that can encode their assignments.

$x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0$
Encoding assignments

Converting $\varphi$ to a graph

Given a formula with $n$ variables, we need a graph with $2^n$ different Hamiltonian paths, that can encode their assignments.

$x_1 = 1, x_2 = 0, x_3 = 0, x_4 = 0$
Encoding assignments

Converting $\varphi$ to a graph

Given a formula with $n$ variables, we need a graph with $2^n$ different Hamiltonian paths, that can encode their assignments.

$x_1 = 0, x_2 = 1, x_3 = 0, x_4 = 0$
Encoding assignments

Converting \( \varphi \) to a graph

Given a formula with \( n \) variables, we need a graph with \( 2^n \) different Hamiltonian paths, that can encode their assignments.

\[
x_1 = 1, \ x_2 = 1, \ x_3 = 0, \ x_4 = 0
\]
Encoding assignments

Converting \( \varphi \) to a graph

Given a formula with \( n \) variables, we need a graph with \( 2^n \) different Hamiltonian paths, that can encode their assignments.

\[
\begin{align*}
\ x_1 &= 0, \ x_2 = 0, \ x_3 = 1, \ x_4 = 0 
\end{align*}
\]
Encoding assignments

Converting $\varphi$ to a graph

Given a formula with $n$ variables, we need a graph with $2^n$ different Hamiltonian paths, that can encode their assignments.

$x_1 = 1$, $x_2 = 0$, $x_3 = 1$, $x_4 = 0$
Encoding assignments

Converting $\varphi$ to a graph

Given a formula with $n$ variables, we need a graph with $2^n$ different Hamiltonian paths, that can encode their assignments.

$x_1 = 0, x_2 = 1, x_3 = 1, x_4 = 0$
Encoding assignments
Converting $\varphi$ to a graph

Given a formula with $n$ variables, we need a graph with $2^n$ different Hamiltonian paths, that can encode their assignments.

$x_1 = 1, x_2 = 1, x_3 = 1, x_4 = 01$
Encoding assignments

Converting $\varphi$ to a graph

Given a formula with $n$ variables, we need a graph with $2^n$ different Hamiltonian paths, that can encode their assignments.

$x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 1$
Encoding assignments

Converting $\varphi$ to a graph

Given a formula with $n$ variables, we need a graph with $2^n$ different Hamiltonian paths, that can encode their assignments.

$x_1 = 1, x_2 = 0, x_3 = 0, x_4 = 1$
Encoding assignments

Converting $\varphi$ to a graph

Given a formula with $n$ variables, we need a graph with $2^n$ different Hamiltonian paths, that can encode their assignments.

$x_1 = 0$, $x_2 = 1$, $x_3 = 0$, $x_4 = 1$
Encoding assignments

Converting $\varphi$ to a graph

Given a formula with $n$ variables, we need a graph with $2^n$ different Hamiltonian paths, that can encode their assignments.

$x_1 = 1, x_2 = 1, x_3 = 0, x_4 = 1$
Encoding assignments

Converting $\varphi$ to a graph

Given a formula with $n$ variables, we need a graph with $2^n$ different Hamiltonian paths, that can encode their assignments.

$x_1 = 0, x_2 = 0, x_3 = 1, x_4 = 1$
Encoding assignments

Converting $\varphi$ to a graph

Given a formula with $n$ variables, we need a graph with $2^n$ different Hamiltonian paths, that can encode their assignments.

$x_1 = 1, x_2 = 0, x_3 = 1, x_4 = 1$
Encoding assignments

Converting $\varphi$ to a graph

Given a formula with $n$ variables, we need a graph with $2^n$ different Hamiltonian paths, that can encode their assignments.

$x_1 = 0, x_2 = 1, x_3 = 1, x_4 = 1$
Encoding assignments

Converting $\varphi$ to a graph

Given a formula with $n$ variables, we need a graph with $2^n$ different Hamiltonian paths, that can encode their assignments.

$x_1 = 1, x_2 = 1, x_3 = 1, x_4 = 1$
Encoding assignments

Converting $\varphi$ to a graph

Given a formula with $n$ variables, we need a graph with $2^n$ different Hamiltonian paths, that can encode their assignments.
THE END

... 

(for now)