20.4
Correctness of the MST algorithms
20.4.1
Safe edges must be in the MST
Correctness of MST Algorithms

1. Many different MST algorithms
2. All of them rely on some basic properties of MSTs, in particular the Cut Property to be seen shortly.
Key Observation: Cut Property

**Lemma 20.1.**

*If* $e$ *is a safe edge then every minimum spanning tree contains* $e$.

**Proof.**

1. Suppose (for contradiction) $e$ is not in MST $T$.
2. Since $e$ is safe there is an $S \subset V$ such that $e$ is the unique min cost edge crossing $S$.
3. Since $T$ is connected, there must be some edge $f$ with one end in $S$ and the other in $V \setminus S$.
4. Since $c_f > c_e$, $T' = (T \setminus \{f\}) \cup \{e\}$ is a spanning tree of lower cost! Error: $T'$ may not be a spanning tree!!
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Problematic example. $S = \{1, 2, 7\}$, $e = (7, 3)$, $f = (1, 6)$. $T - f + e$ is not a spanning tree.

(A) Consider adding the edge $f$. 

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Error in Proof: Example

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(B) It is safe because it is the cheapest edge in the cut.
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(C) Lets throw out the edge \( e \) currently in the spanning tree which is more expensive than \( f \) and is in the same cut. Put it \( f \) instead...
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(C) Lets throw out the edge $e$ currently in the spanning tree which is more expensive than $f$ and is in the same cut. Put it $f$ instead...
(D) New graph of selected edges is not a tree anymore. BUG.
Proof of Cut Property

Proof.

1. Suppose $e = (v, w)$ is not in MST $T$ and $e$ is min weight edge in cut $(S, V \setminus S)$. Assume $v \in S$.

2. $T$ is spanning tree: there is a unique path $P$ from $v$ to $w$ in $T$.

3. Let $w'$ be the first vertex in $P$ belonging to $V \setminus S$; let $v'$ be the vertex just before it on $P$, and let $e' = (v', w')$.

4. $T' = (T \setminus \{e'\}) \cup \{e\}$ is spanning tree of lower cost. (Why?)
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Proof of Cut Property (contd)

**Observation 20.2.**

\[ T' = (T \setminus \{e'\}) \cup \{e\} \text{ is a spanning tree.} \]

**Proof.**

\[ T' \text{ is connected.} \]

Removed \( e' = (v', w') \) from \( T \) but \( v' \) and \( w' \) are connected by the path \( P = f + e \) in \( T' \). Hence \( T' \) is connected if \( T \) is.

\[ T' \text{ is a tree} \]

\( T' \) is connected and has \( n - 1 \) edges (since \( T \) had \( n - 1 \) edges) and hence \( T' \) is a tree.
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THE END

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(for now)