19.6.3
Proving optimality of earliest finish time
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Instead we will show that $|O| = |X|$. 
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![Diagram of jobs and conflicts](image-url)
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Claim 19.3.

Let \( i \) be the first interval picked by Greedy into the solution. Let \( O \) be the optimal solution.

If \( i \not\in O \), there is exactly one interval \( j_1 \in O \) that conflicts with \( i \).

Proof.

1. No \( j \in O \) conflicts \( i \) \( \implies \) \( O \) is not opt!
2. Suppose \( j_1, j_2 \in O \) such that \( j_1 \neq j_2 \) and both \( j_1 \) and \( j_2 \) conflict with \( i \).
3. Since \( i \) has earliest finish time, \( j_1 \) and \( i \) overlap at \( f(i) \).
4. For same reason \( j_2 \) also overlaps with \( i \) at \( f(i) \).
5. Implies that \( j_1, j_2 \) overlap at \( f(i) \) but intervals in \( O \) cannot overlap.
Proof of Optimality: Key Lemma

**Lemma 19.4.**

\( i_1 \) be first interval picked by Greedy. There exists an optimum solution that contains \( i_1 \).

**Proof.**

Let \( O \) be an arbitrary optimum solution. If \( i_1 \in O \) we are done.

By **Claim 19.3** ...

1. Exists exactly one \( j_1 \in O \) conflicting with \( i_1 \).
2. Form a new set \( O' \) by removing \( j_1 \) from \( O \) and adding \( i_1 \), that is \( O' = (O - \{j_1\}) \cup \{i_1\} \).
3. From claim, \( O' \) is a feasible solution (no conflicts).
4. Since \( |O'| = |O| \), \( O' \) is also an optimum solution and it contains \( i_1 \).
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Proof of Optimality of Earliest Finish Time First

Proof by Induction on number of intervals.

Base Case: $n = 1$. Trivial since Greedy picks one interval.

Induction Step: Assume theorem holds for $i < n$.

Let $K$ be an input (i.e., instance) with $n$ intervals

$i_1 \leftarrow$ First interval picked by greedy algorithm.

$K' \leftarrow$ The result of removing $i_1$ and all conflicting intervals from $K$.

$|K'| = |K| - 1.$

$G(K), G(K')$: Solution produced by Greedy on $K$ and $K'$, respectively.

Lemma 19.4 $\implies$ optimum solution $O$ to $K$ with $i_1 \in O$.

Let $O' = O - \{i_1\}$. $O'$ is a solution to $K'$.

$$|G(K)| = 1 + |G(K')|$$

$$\geq 1 + |O'|$$

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from Greedy description

By induction, $G(I')$ is optimum for $I'$)
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