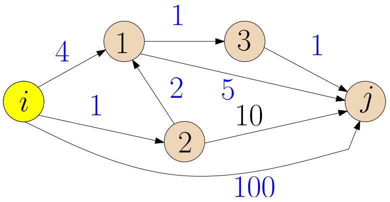
18.4.2
All Pairs Shortest Paths: A recursive solution

All-Pairs: Recursion on index of intermediate nodes

1. Number vertices arbitrarily as $v_1, v_2, \ldots, v_n$
2. $dist(i, j, k)$: length of shortest walk from v_i to v_j among all walks in which the largest index of an intermediate node is at most k (could be $-\infty$ if there is a negative length cycle).



$$dist(i, j, 0) = 100$$
$$dist(i, j, 1) = 9$$
$$dist(i, j, 2) = 8$$
$$dist(i, j, 3) = 5$$

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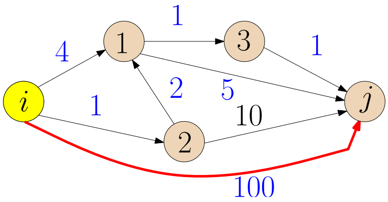

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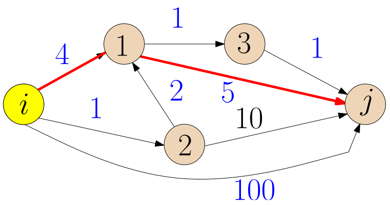

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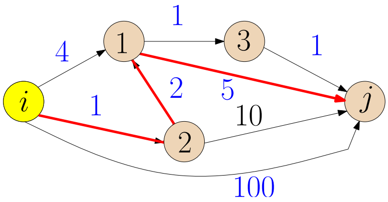

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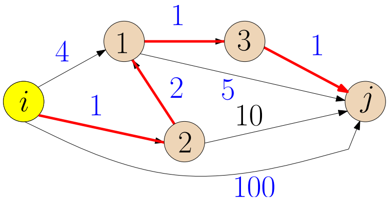

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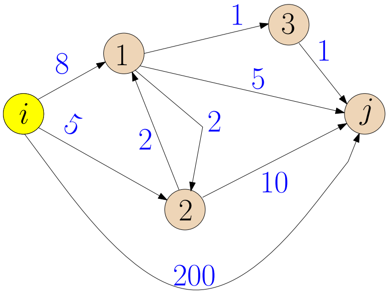
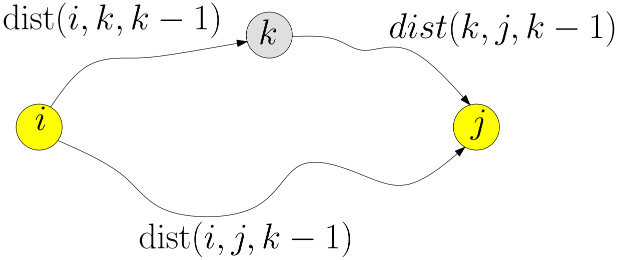
For the following graph, **dist(i, j, 2)** is...



- **9**
- **10**
- **11**
- **12**
- **15**

All-Pairs: Recursion on index of intermediate nodes



$$\mathbf{dist}(i,j,k) = \min \begin{cases} \mathbf{dist}(i,j,k-1) \\ \mathbf{dist}(i,k,k-1) + \mathbf{dist}(k,j,k-1) \end{cases}$$

Base case: $\mathbf{dist}(i,j,0) = \ell(i,j)$ if $(i,j) \in E$, otherwise ∞

Correctness: If $i \to j$ shortest walk goes through k then k occurs only once on the path — otherwise there is a negative length cycle.

All-Pairs: Recursion on index of intermediate nodes

If i can reach k and k can reach j and $dist(k, k, k-1) < 0$ then G has a negative length cycle containing k and $dist(i, j, k) = -\infty$.

Recursion below is valid only if $dist(k, k, k-1) \geq 0$. We can detect this during the algorithm or wait till the end.

$$dist(i, j, k) = \min \begin{cases} dist(i, j, k-1) \\ dist(i, k, k-1) + dist(k, j, k-1) \end{cases}$$

THE END

...

(for now)