18.2.4
Bellman-Ford: Detecting negative cycles

Correctness: detecting negative length cycle

Lemma 18.5.

Suppose G has a negative cycle C reachable from s. Then there is some node $v \in C$ such that $d(v, n) < d(v, n-1)$.

Proof.

Suppose not. Let $C = v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_h \rightarrow v_1$ be negative length cycle reachable from s. $d(v_i, n-1)$ is finite for $1 \leq i \leq h$ since C is reachable from s. By assumption $d(v, n) \geq d(v, n-1)$ for all $v \in C$; implies no change in nth iteration; $d(v_i, n-1) = d(v_i, n)$ for $1 \leq i \leq h$. This means $d(v_i, n-1) \leq d(v_{i-1}, n-1) + \ell(v_{i-1}, v_i)$ for $2 \leq i \leq h$ and $d(v_1, n-1) \leq d(v_n, n-1) + \ell(v_n, v_1)$. Adding up all these inequalities results in the inequality $0 \leq \ell(C)$ which contradicts the assumption that $\ell(C) < 0$. □

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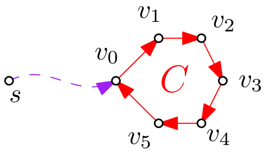
Proof of **Lemma 18.5** in more detail...



$$d(v_1, n) \leq d(v_0, n-1) + \ell(v_0, v_1)$$
$$d(v_2, n) \leq d(v_1, n-1) + \ell(v_1, v_2)$$
$$\dots$$
$$d(v_i, n) \leq d(v_{i-1}, n-1) + \ell(v_{i-1}, v_i)$$
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$$d(v_k, n) \leq d(v_{k-1}, n-1) + \ell(v_{k-1}, v_k)$$
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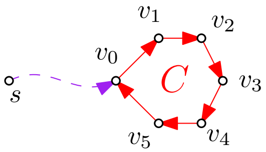

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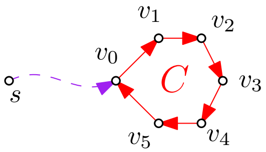

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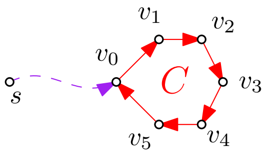
$$\sum_{i=0}^{k} d(v_i, n) \leq \sum_{i=0}^{k} d(v_i, n) + \sum_{i=1}^{k} \ell(v_{i-1}, v_i) + \ell(v_k, v_0)$$

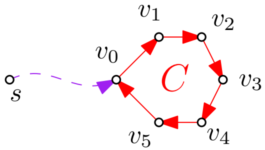
Proof of **Lemma 18.5** in more detail...



$$\sum_{i=0}^{k} d(v_i, n) \leq \sum_{i=0}^{k} d(v_i, n) + \sum_{i=1}^{k} \ell(v_{i-1}, v_i) + \ell(v_k, v_0)$$

$$0 \leq \sum_{i=1}^{k} \ell(v_{i-1}, v_i) + \ell(v_k, v_0).$$

Proof of **Lemma 18.5** in more detail...



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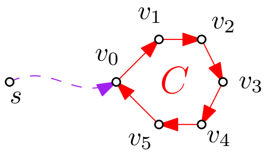
$$0 \leq \sum_{i=1}^{k} \ell(v_{i-1}, v_i) + \ell(v_k, v_0) = \mathrm{len}(C)\,.$$

Proof of **Lemma 18.5** in more detail...



$$\sum_{i=0}^{k} d(v_i, n) \le \sum_{i=0}^{k} d(v_i, n) + \sum_{i=1}^{k} \ell(v_{i-1}, v_i) + \ell(v_k, v_0)$$

$$0 \le \sum_{i=1}^{k} \ell(v_{i-1}, v_i) + \ell(v_k, v_0) = \mathrm{len}(C)\,.$$

C is a not a negative cycle. Contradiction. □

Negative cycles can not hide

Lemma 18.4 restated

If G does not has a negative length cycle reachable from $s \implies \forall v$:
$d(v, n) = d(v, n-1)$.

Also, $d(v, n-1)$ is the length of the shortest path between s and v.

Lemma 18.4 and **Lemma 18.5** put together are the following:

Lemma 18.6.

G has a negative length cycle reachable from $s \iff$ there is some node v such that $d(v, n) < d(v, n-1)$.

Bellman-Ford: Negative Cycle Detection

The official final version

```
for each u ∈ V do
    d(u) ← ∞
d(s) ← 0

for k = 1 to n − 1 do
    for each v ∈ V do
        for each edge (u, v) ∈ in(v) do
            d(v) = min{d(v), d(u) + ℓ(u, v)}
(* One more iteration to check if distances change *)
for each v ∈ V do
    for each edge (u, v) ∈ in(v) do
        if (d(v) > d(u) + ℓ(u, v))
            Output ``Negative Cycle''

for each v ∈ V do
    dist(s, v) ← d(v)
```

THE END

...

(for now)