18.2.3.1 Correctness of the Bellman-Ford Algorithm
Bellman-Ford Algorithm: Modified for analysis

for each $u \in V$ do
    $d(u, 0) \leftarrow \infty$
    $d(s, 0) \leftarrow 0$

for $k = 1$ to $n$ do
    for each $v \in V$ do
        $d(v, k) \leftarrow d(v, k - 1)$
        for each edge $(u, v) \in in(v)$ do
            $d(v, k) = \min\{d(v, k), d(u, k - 1) + \ell(u, v)\}$

for each $v \in V$ do
    $\text{dist}(s, v) \leftarrow d(v, n - 1)$
Lemma 18.3.

For each $v$, $d(v, k)$ is the length of a shortest walk from $s$ to $v$ with at most $k$ hops.

Proof.

Standard induction (left as exercise).
Bellman-Ford computes the shortest paths correctly

**Lemma 18.4.**

If $G$ does not have a negative length cycle reachable from $s \Rightarrow \forall v$: $d(v, n) = d(v, n - 1)$.

Also, $d(v, n - 1)$ is the length of the shortest path between $s$ and $v$.

**Proof.**

Shortest walk from $s$ to reachable vertex is a path [not repeated vertex] (otherwise $\exists$ neg cycle).

A path has at most $n - 1$ edges.

$\Rightarrow$ Len shortest walk from $s$ to $v$ with at most $n - 1$ edges

$= \text{Len shortest walk from } s \text{ to } v$

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By **Lemma 18.3**: $d(v, n) = d(v, n - 1) = \text{dist}(s, v)$, for all $v$.  

□
Bellman-Ford computes the shortest paths correctly

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Bellman-Ford computes the shortest paths correctly

**Lemma 18.4.**

If $G$ does not have a negative length cycle reachable from $s \implies \forall v$: $d(v, n) = d(v, n - 1)$.

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Bellman-Ford computes the shortest paths correctly

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Also, \( d(v, n - 1) \) is the length of the shortest path between \( s \) and \( v \).

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THE END
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(for now)