18.1.4
Applications of shortest path for negative weights on edges
Why negative lengths?

Several Applications

1. Shortest path problems useful in modeling many situations — in some negative lengths are natural
2. Negative length cycle can be used to find arbitrage opportunities in currency trading
3. Important sub-routine in algorithms for more general problem: minimum-cost flow
Negative cycles
Application to Currency Trading

Currency Trading

**Input:** \( n \) currencies and for each ordered pair \((a, b)\) the exchange rate for converting one unit of \( a \) into one unit of \( b \).

**Questions:**

1. Is there an arbitrage opportunity?
2. Given currencies \( s, t \) what is the best way to convert \( s \) to \( t \) (perhaps via other intermediate currencies)?

Concrete example:

1. 1 Chinese Yuan = 0.1116 Euro
2. 1 Euro = 1.3617 US dollar
3. 1 US Dollar = 7.1 Chinese Yuan.

Thus, if exchanging 1 $ → Yuan → Euro → $, we get: \( 0.1116 \times 1.3617 \times 7.1 = 1.07896\) $.
Reducing Currency Trading to Shortest Paths

Observation: If we convert currency $i$ to $j$ via intermediate currencies $k_1, k_2, \ldots, k_h$ then one unit of $i$ yields $\text{exch}(i, k_1) \times \text{exch}(k_1, k_2) \ldots \times \text{exch}(k_h, j)$ units of $j$.

Create currency trading directed graph $G = (V, E)$:
1. For each currency $i$ there is a node $v_i \in V$
2. $E = V \times V$: an edge for each pair of currencies
3. edge length $\ell(v_i, v_j) = -\log(\text{exch}(i, j))$ can be negative

Exercise: Verify that
1. There is an arbitrage opportunity if and only if $G$ has a negative length cycle.
2. The best way to convert currency $i$ to currency $j$ is via a shortest path in $G$ from $i$ to $j$. If $d$ is the distance from $i$ to $j$ then one unit of $i$ can be converted into $2^d$ units of $j$. 
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Math recall - relevant information

1. \( \log(\alpha_1 \times \alpha_2 \times \cdots \times \alpha_k) = \log \alpha_1 + \log \alpha_2 + \cdots + \log \alpha_k. \)

2. \( \log x > 0 \) if and only if \( x > 1 \).
THE END

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(for now)