15.4

Directed Graphs and Decomposition
Directed Graphs

Definition

A directed graph \( G = (V, E) \) consists of

1. set of vertices/nodes \( V \) and
2. a set of edges/arcs \( E \subseteq V \times V \).

An edge is an ordered pair of vertices. \((u, v)\) different from \((v, u)\).
Examples of Directed Graphs

In many situations relationship between vertices is asymmetric:

1. Road networks with one-way streets.

2. Web-link graph: vertices are web-pages and there is an edge from page $p$ to page $p'$ if $p$ has a link to $p'$. Web graphs used by Google with PageRank algorithm to rank pages.

3. Dependency graphs in variety of applications: link from $x$ to $y$ if $y$ depends on $x$. Make files for compiling programs.

4. Program Analysis: functions/procedures are vertices and there is an edge from $x$ to $y$ if $x$ calls $y$. 
Directed Graph Representation

Graph $G = (V, E)$ with $n$ vertices and $m$ edges:


2. **Adjacency Lists**: for each node $u$, $Out(u)$ (also referred to as $Adj(u)$) and $In(u)$ store out-going edges and in-coming edges from $u$.

Default representation is adjacency lists.
A Concrete Representation for Directed Graphs

Concrete representation discussed previously for undirected graphs easily extends to directed graphs.

Array of edges $E$

Array of adjacency lists

List of edges (indices) that are incident to $v_i$
Directed Connectivity

Given a graph $G = (V, E)$:

1. A **(directed) path** is a sequence of distinct vertices $v_1, v_2, \ldots, v_k$ such that $(v_i, v_{i+1}) \in E$ for $1 \leq i \leq k - 1$. The length of the path is $k - 1$ and the path is from $v_1$ to $v_k$.
   
   By convention, a single node $u$ is a path of length $0$.

2. A cycle is a sequence of distinct vertices $v_1, v_2, \ldots, v_k$ such that $(v_i, v_{i+1}) \in E$ for $1 \leq i \leq k - 1$ and $(v_k, v_1) \in E$.
   
   By convention, a single node $u$ is not a cycle.

3. A vertex $u$ can reach $v$ if there is a path from $u$ to $v$. Alternatively, $v$ can be reached from $u$.

4. Let $rch(u)$ be the set of all vertices reachable from $u$. 
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Connectivity contd

Asymmetricity:  $D$ can reach $B$ but $B$ cannot reach $D$

Questions:
1. Is there a notion of connected components?
2. How do we understand connectivity in directed graphs?
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1. Is there a notion of connected components?
2. How do we understand connectivity in directed graphs?
THE END

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(for now)