12.4
Longest Increasing Subsequence

Sequences

Definition

Sequence: an ordered list $a_1, a_2, \ldots, a_n$. **Length** of a sequence is number of elements in the list.

Definition

$a_{i_1}, \ldots, a_{i_k}$ is a **subsequence** of $a_1, \ldots, a_n$ if $1 \leq i_1 < i_2 < \ldots < i_k \leq n$.

Definition

A sequence is **increasing** if $a_1 < a_2 < \ldots < a_n$. It is **non-decreasing** if $a_1 \leq a_2 \leq \ldots \leq a_n$. Similarly **decreasing** and **non-increasing**.

Sequences

Example...

Example

1. Sequence: $6, 3, 5, 2, 7, 8, 1, 9$
2. Subsequence of above sequence: $5, 2, 1$
3. Increasing sequence: $3, 5, 9, 17, 54$
4. Decreasing sequence: $34, 21, 7, 5, 1$
5. Increasing subsequence of the first sequence: $2, 7, 9$.

Longest Increasing Subsequence Problem

Input A sequence of numbers $a_1, a_2, \ldots, a_n$

Goal Find an **increasing subsequence** $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ of maximum length

Example

1. Sequence: 6, 3, 5, 2, 7, 8, 1
2. Increasing subsequences: 6, 7, 8 and 3, 5, 7, 8 and 2, 7 etc
3. Longest increasing subsequence: 3, 5, 7, 8

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Naïve Enumeration

Assume $a_1, a_2, \ldots, a_n$ is contained in an array A

```
algLISNaive(A[1..n]):
    max = 0
    for each subsequence B of A do
        if B is increasing and |B| > max then
            max = |B|

    Output max
```

Running time: $O(n2^n)$.

2^n subsequences of a sequence of length n and $O(n)$ time to check if a given sequence is increasing.

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Recursive Approach: Take 1

LIS: Longest increasing subsequence

Can we find a recursive algorithm for LIS?

$\mathrm{LIS}(A[1..n])$:

1. Case 1: Does not contain $A[n]$ in which case $\mathrm{LIS}(A[1..n]) = \mathrm{LIS}(A[1..(n-1)])$
2. Case 2: contains $A[n]$ in which case $\mathrm{LIS}(A[1..n])$ is not so clear.

Observation

For second case we want to find a subsequence in $A[1..(n-1)]$ that is restricted to numbers less than $A[n]$. This suggests that a more general problem is **LIS_smaller**$(A[1..n], x)$ *which gives the longest increasing subsequence in A where each number in the sequence is less than x.*

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Recursive Approach

LIS_smaller$(A[1..n], x)$: length of longest increasing subsequence in $A[1..n]$ with all numbers in subsequence less than x

```
LIS_smaller(A[1..n], x):
    if (n = 0) then return 0
    m = LIS_smaller(A[1..(n − 1)], x)
    if (A[n] < x) then
        m = max(m, 1 + LIS_smaller(A[1..(n − 1)], A[n]))
    Output m
```

```
LIS(A[1..n]):
        return LIS_smaller(A[1..n], ∞)
```

Example

Sequence: $\boldsymbol{A}[1..7] = 6, 3, 5, 2, 7, 8, 1$

THE END

...

(for now)