11.2
Multiplication using Divide and Conquer

Divide and Conquer

Assume n is a power of 2 for simplicity and numbers are in decimal.

Split each number into two numbers with equal number of digits

1. $b = b_{n-1}b_{n-2}\ldots b_0$ and $c = c_{n-1}c_{n-2}\ldots c_0$
2. $b = b_{n-1}\ldots b_{n/2}0\ldots 0 + b_{n/2-1}\ldots b_0$
3. $b(x) = b_L x + b_R$, where $x = 10^{n/2}$, $b_L = b_{n-1}\ldots b_{n/2}$ and $b_R = b_{n/2-1}\ldots b_0$
4. Similarly $c(x) = c_L x + c_R$ where $c_L = c_{n-1}\ldots c_{n/2}$ and $c_R = c_{n/2-1}\ldots c_0$

Example

$$1234 \times 5678 = (12x + 34) \times (56x + 78) \qquad \text{for} \quad x = 100.$$
$$= 12 \cdot 56 \cdot x^2 + (12 \cdot 78 + 34 \cdot 56)x + 34 \cdot 78.$$

$$\begin{aligned} 1234 \times 5678 &= (100 \times 12 + 34) \times (100 \times 56 + 78) \\ &= 10000 \times 12 \times 56 \\ &\quad +100 \times (12 \times 78 + 34 \times 56) \\ &\quad +34 \times 78 \end{aligned}$$

Divide and Conquer for multiplication

Assume n is a power of 2 for simplicity and numbers are in decimal.

1. $b = b_{n-1} b_{n-2} \ldots b_0$ and $c = c_{n-1} c_{n-2} \ldots c_0$
2. $b \equiv b(x) = b_L x + b_R$
   where $x = 10^{n/2}$, $b_L = b_{n-1} \ldots b_{n/2}$ and $b_R = b_{n/2-1} \ldots b_0$
3. $c \equiv c(x) = c_L x + c_R$ where $c_L = c_{n-1} \ldots c_{n/2}$ and $c_R = c_{n/2-1} \ldots c_0$

Therefore, for $x = 10^{n/2}$, we have

$$\begin{aligned} bc = b(x)c(x) &= (b_L x + b_R)(c_L x + c_R) \\ &= b_L c_L x^2 + (b_L c_R + b_R c_L)x + b_R c_R \\ &= 10^n b_L c_L + 10^{n/2}(b_L c_R + b_R c_L) + b_R c_R \end{aligned}$$

Divide and Conquer for multiplication

Assume n is a power of 2 for simplicity and numbers are in decimal.

1. $b = b_{n-1}b_{n-2}\ldots b_0$ and $c = c_{n-1}c_{n-2}\ldots c_0$
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3. $c \equiv c(x) = c_L x + c_R$ where $c_L = c_{n-1}\ldots c_{n/2}$ and $c_R = c_{n/2-1}\ldots c_0$

Therefore, for $x = 10^{n/2}$, we have

$$
\begin{aligned}
bc &= b(x)c(x) = (b_L x + b_R)(c_L x + c_R) \\
&= b_L c_L x^2 + (b_L c_R + b_R c_L)x + b_R c_R \\
&= 10^n b_L c_L + 10^{n/2}(b_L c_R + b_R c_L) + b_R c_R
\end{aligned}
$$

Time Analysis

$$bc = 10^n b_L c_L + 10^{n/2}(b_L c_R + b_R c_L) + b_R c_R$$

$\mathbf{4}$ recursive multiplications of number of size $n/2$ each plus 4 additions and left shifts (adding enough 0's to the right)

$$T(n) = 4T(n/2) + O(n) \qquad T(1) = O(1)$$

$T(n) = \Theta(n^2)$. No better than grade school multiplication!

Time Analysis

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4 recursive multiplications of number of size $n/2$ each plus 4 additions and left shifts (adding enough 0's to the right)

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$T(n) = \Theta(n^2)$. No better than grade school multiplication!

THE END

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(for now)