9.4

Unrecognizable
**Definition**

Language $L$ is **TM decidable** if there exists $M$ that always stops, such that $L(M) = L$.

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Language $L$ is **TM recognizable** if there exists $M$ that stops on some inputs, such that $L(M) = L$.

**Theorem (Halting)**

$A_{TM} = \{ \langle M, w \rangle \mid M$ is a TM and $M$ accepts $w \}$. is TM recognizable, but not decidable.
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**Theorem (Halting)**

$$A_{TM} = \{ \langle M, w \rangle \mid M \text{ is a TM and } M \text{ accepts } w \}.$$ is TM recognizable, but not decidable.
Lemma

If $L$ and $\overline{L} = \Sigma^* \setminus L$ are both TM recognizable, then $L$ and $\overline{L}$ are decidable.

Proof.

$M$: TM recognizing $L$.

$M_c$: TM recognizing $\overline{L}$.

Given input $x$, using UTM simulating running $M$ and $M_c$ on $x$ in parallel. One of them must stop and accept. Return result.

$\implies L$ is decidable.
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\[ \Rightarrow L \text{ is decidable.} \]
Complement language for $A_{TM}$

$$A_{TM} = \sum^* \setminus \set{\langle M, w \rangle \mid M \text{ is a TM and } M \text{ accepts } w}.$$  

But don’t really care about invalid inputs. So, really:

$$A_{TM} = \set{\langle M, w \rangle \mid M \text{ is a TM and } M \text{ does not accept } w}.$$
Complement language for $A_{TM}$

$$A_{TM} = \Sigma^* \setminus \{ \langle M, w \rangle \mid M \text{ is a TM and } M \text{ accepts } w \}.$$

But don’t really care about invalid inputs. So, really:

$$\overline{A_{TM}} = \{ \langle M, w \rangle \mid M \text{ is a TM and } M \text{ does not accept } w \}.$$
Theorem

The language

\[ \overline{A_{TM}} = \left\{ \langle M, w \rangle \mid M \text{ is a TM and } M \text{ does not accept } w \right\} \]

is not TM recognizable.

Proof.

If \( A_{TM} \) is TM-recognizable

\[ \implies \] (by Lemma)

\( A_{TM} \) is decidable. A contradiction.
Theorem

The language

$$\overline{A_{TM}} = \left\{ \langle M, w \rangle \mid M \text{ is a TM and } M \text{ does not accept } w \right\}.$$  

is not TM recognizable.

Proof.

$A_{TM}$ is TM-recognizable.

If $A_{TM}$ is TM-recognizable

$$\implies \quad \text{(by Lemma)}$$

$A_{TM}$ is decidable. A contradiction.
Complement language for $A_{TM}$ is not TM-recognizable

**Theorem**

The language

$$A_{TM} = \left\{ \langle M, w \rangle \mid M \text{ is a TM and } M \text{ does not accept } w \right\}.$$ 

is not TM recognizable.

**Proof.**

If $A_{TM}$ is TM-recognizable

$\implies$ (by Lemma)

$A_{TM}$ is decidable. A contradiction.
THE END

...(for now)