9.2

Introduction to the halting theorem
The halting problem

**Halting problem:** Given a program $Q$, if we run it would it stop?

**Q:** Can one build a program $P$, that always stops, and solves the halting problem.

**Theorem (“Halting theorem”)**

*There is no program that always stops and solves the halting problem.*
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**Theorem ("Halting theorem")**

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Intuition, why solving the Halting problem is really hard

Definition

An integer number $n$ is a **weird number** if

- the sum of the proper divisors (including 1 but not itself) of $n$ the number is $> n$,
- no subset of those divisors sums to the number itself.

70 is weird. Its divisors are $1, 2, 5, 7, 10, 14, 35$. $1 + 2 + 5 + 7 + 10 + 14 + 35 = 74$.
No subset of them adds up to 70.

Open question: Are there any odd weird numbers?

Write a program $P$ that tries all odd numbers in order, and check if they are weird. The program stops if it found such number.

If can solve halting problem $\iff$ can resolve this open problem.
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If can solve halting problem \( \iff \) can resolve this open problem.
If you can halt, you can prove or disprove anything...

Consider any math claim $C$.

Prover algorithm $P_C$:

(A) Generate sequence of all possible proofs (sequence of strings) into a pipe/queue.
(B) $\langle p \rangle \leftarrow$ pop top of queue.
(C) Feed $\langle p \rangle$ and $\langle C \rangle$, into a proof verifier (“easy”).
(D) If $\langle p \rangle$ valid proof of $\langle C \rangle$, then stop and accept.
(E) Go to (B).

$P_C$ halts $\iff C$ is true and has a proof.

If halting is decidable, then can decide if any claim in math is true.
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1. Consider any math claim $C$.
2. Prover algorithm $P_C$:
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3. $P_C$ halts $\iff$ $C$ is true and has a proof.
4. If halting is decidable, then can decide if any claim in math is true.
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THE END

... (for now)