Halting, Undecidability, and Maybe Some Complexity

Lecture 9
Tuesday, September 22, 2020
“Young man, in mathematics you don’t understand things. You just get used to them.”
– John von Neumann.
9.1 Cantor’s diagonalization argument
You can not count the real numbers

\[ I = (0, 1). \]
\[ \mathbb{N} = \{1, 2, 3, \ldots\} \text{ the integer numbers} \]

**Claim (Cantor)**

\[ |\mathbb{N}| \neq |I| \]

**Claim (Warm-up)**

\[ |\mathbb{N}| \leq |I| \]

**Proof.**

\[ |\mathbb{N}| \leq |I| \] exists a one-to-one mapping from \( \mathbb{N} \) to \( I \). One such mapping is \( f(i) = 1/i \), which readily implies the claim.
You can not count the real numbers

$I = (0, 1)$. 
$\mathbb{N} = \{1, 2, 3, \ldots\}$ the integer numbers

**Claim (Cantor)**

$|\mathbb{N}| \neq |I|$

**Claim (Warm-up)**

$|\mathbb{N}| \leq |I|$

**Proof.**

$|\mathbb{N}| \leq |I|$ exists a one-to-one mapping from $\mathbb{N}$ to $I$. One such mapping is $f(i) = 1/i$, which readily implies the claim.
You can not count the real numbers II

\( I = (0, 1), \ N = \{1, 2, 3, \ldots\}. \)

Claim (Cantor)

\[ |\mathbb{N}| \neq |I|, \text{ where } I = (0, 1). \]

Proof.

Write every number in \((0, 1)\) in its decimal expansion. E.g.,
\[ 1/3 = 0.33333333333333333333\ldots. \]
Assume that \(|\mathbb{N}| = |I|\). Then there exists a one-to-one mapping \( f : \mathbb{N} \rightarrow I \). Let \( \beta_i \) be the \( i \)th digit of \( f(i) \in (0, 1) \).

\[ d_i = \text{ any number in } \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \setminus \{d_{i-1}, \beta_i\} \]

\[ D = 0.d_1d_2d_3\ldots \in (0, 1). \]

\( D \) is a well defined unique number in \((0, 1)\),
But there is no \( j \) such that \( f(j) = D \). A contradiction.
You can not count the real numbers II

$I = (0, 1), \mathbb{N} = \{1, 2, 3, \ldots\}.$

Claim (Cantor)

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Write every number in $(0, 1)$ in its decimal expansion. E.g.,

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But there is no $j$ such that $f(j) = D$. A contradiction.
The matrix...

<table>
<thead>
<tr>
<th></th>
<th>$f(1)$</th>
<th>$f(2)$</th>
<th>$f(3)$</th>
<th>$f(4)$</th>
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<td>1</td>
<td>$\beta_1 = 1$</td>
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<td>0</td>
<td>0</td>
<td>0</td>
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<tr>
<td>2</td>
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<td>$\beta_2 = 1$</td>
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<td>3</td>
<td>1</td>
<td>0</td>
<td>$\beta_3 = 1$</td>
<td>1</td>
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<tr>
<td>4</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>$\beta_4 = 0$</td>
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$\implies \forall i \beta_i \neq d_i.$
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$\implies \forall i \beta_i \neq d_i.$

$D = 0.23232323\ldots$

$D$ can not be the $i$ column, because $\beta_i \neq d_i.$
The matrix...

\[
\begin{array}{cccccc}
 & f(1) & f(2) & f(3) & f(4) & \\
1 & 1 & 1 & 0 & 0 & \ldots \\
2 & 0 & 1 & 0 & 1 & \ldots \\
3 & 1 & 0 & 1 & 1 & \ldots \\
4 & 0 & 1 & 0 & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
\]

\(d_i = \) any number in \(\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \setminus \{d_i-1, \beta_i\}\)

\(\implies \forall i \beta_i \neq d_i.\)

\(D = 0.23232323\ldots\)

\(D\) can not be the \(i\) column, because \(\beta_i \neq d_i.\)

But \(D\) can not be in the matrix...
The liar paradox

*When one day an expedition was sent to the spatial coordinates that Voojagig had claimed for this planet they discovered only a small asteroid inhabited by a solitary old man who claimed repeatedly that nothing was true, though he was later discovered to be lying.*

– The Hitchhiker Guide to the Galaxy

1. The liar’s paradox: This sentence is false.
2. Related to Russell’s paradox.
3. Omnipotence paradox: Can [an omnipotent being] create a stone so heavy that it cannot lift it?
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THE END

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(for now)