7.8
Supplemental: Why $a^n b^n c^n$ is not CFL
You are bound to repeat yourself...

$L = \{a^n b^n c^n \mid n \geq 0\}$.

1. For the sake of contradiction assume that there exists a grammar:
   \[ G \text{ a CFG for } L. \]

2. \(T_i\): minimal parse tree in \(G\) for \(a^i b^i c^i\).

3. \(h_i = \text{height}(T_i)\): Length of longest path from root to leaf in \(T_i\).

4. For any integer \(t\), there must exist an index \(j(t)\), such that \(h_{j(t)} > t\).

5. There an index \(j\), such that \(h_j > \left(2 \ast \# \text{ variables in } G\right)\).
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Repetition in the parse tree...
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$$xyzvw = a^i b^i c^i$$
Repetition in the parse tree...

\[ xyzvw = a^i b^j c^j \implies xy^2 z v^2 w \in L \]
We know:

\[ xyzvw = a^j b^j c^j \]

\[ |y| + |v| > 0. \]

We proved that \( \tau = xy^2zv^2w \in L \).

If \( y \) contains both \( a \) and \( b \), then, \( \tau = \ldots a \ldots b \ldots a \ldots b \ldots \).

Impossible, since \( \tau \in L = \{a^n b^n c^n \mid n \geq 0\} \).

Similarly, not possible that \( y \) contains both \( b \) and \( c \).

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If \( y \) contains only \( a \)s, and \( v \) contains only \( b \)s, then… \( \#(a)(\tau) \neq \#(c)(\tau) \).

Not possible.

Similarly, not possible that \( y \) contains only \( a \)s, and \( v \) contains only \( c \)s.

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Must be that \( \tau \not\in L \). A contradiction.
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We conclude...

**Lemma**

The language $L = \{ a^n b^n c^n \mid n \geq 0 \}$ is not CFL (i.e., there is no CFG for it).