6.3.1 Exponential gap in number of states between DFA and NFA sizes
Exponential gap between NFA and DFA size

$L_4 = \{ w \in \{0, 1\}^* \mid w \text{ has a } 1 \text{ located } 4 \text{ positions from the end}\}$
Exponential gap between NFA and DFA size

\( L_k = \{ w \in \{0, 1\}^* |\ w \ has \ a \ 1 \ k \ positions \ from \ the \ end \} \)

Recall that \( L_k \) is accepted by a NFA \( N \) with \( k + 1 \) states.

**Theorem**

*Every DFA that accepts \( L_k \) has at least \( 2^k \) states.*

**Claim**

\( F = \{ w \in \{0, 1\}^* : |w| = k \} \) is a fooling set of size \( 2^k \) for \( L_k \).

**Why?**

- Suppose \( a_1a_2\ldots a_k \) and \( b_1b_2\ldots b_k \) are two distinct bitstrings of length \( k \)
- Let \( i \) be first index where \( a_i \neq b_i \)
- \( y = 0^{k-i-1} \) is a distinguishing suffix for the two strings
Exponential gap between NFA and DFA size

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How do we pick a fooling set $\mathcal{F}$?

- If $x, y$ are in $\mathcal{F}$ and $x \neq y$ they should be distinguishable! Of course.
- All strings in $\mathcal{F}$ except maybe one should be prefixes of strings in the language $L$. For example if $L = \{0^k1^k \mid k \geq 0\}$ do not pick 1 and 10 (say). Why?
THE END

... (for now)