1.2 Countable sets, countably infinite sets, and languages
A set $X$ is countable, if its elements can be counted. There exists an injective mapping from $X$ to natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$.

Example
All finite sets are countable: $\{aba, ima, saba, safta, uma, upa\}$.

Example
$\mathbb{N} \times \mathbb{N} = \{(i, j) \mid i, j \in \mathbb{N}\}$ is countable.

Proof: $f(i, j) = 2^i 3^j$. 
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A set $X$ is countably infinite (countable and infinite) if there is a bijection $f$ between the natural numbers and $X$.

Alternatively: $X$ is countably infinite if $X$ is an infinite set and there enumeration of elements of $X$. 
The set of all strings is countable

**Theorem**

$\Sigma^*$ *is countable for any finite* $\Sigma$.

Enumerate strings in order of increasing length and for each given length enumerate strings in dictionary order (based on some fixed ordering of $\Sigma$).

Example: $\{0, 1\}^* = \{\epsilon, 0, 1, 00, 01, 10, 11, 000, 001, 010, \ldots\}$.

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Exercise 1

Question: Is $\Sigma^* \times \Sigma^* = \{(x, y) \mid x, y \in \Sigma^*\}$ countable?

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Exercise II

Answer the following questions taking $\Sigma = \{0, 1\}$.

1. Is a finite set countable?
2. $X$ is countable, and the set $Y \subseteq X$, then is the set $Y$ countable?
3. If $X$ and $Y$ are countable, is $X \setminus Y$ countable?
4. Are all infinite sets countably infinite?
5. If $X_i$ is a countable infinite set, for $i = 1, \ldots, 700$, is $\bigcup_i X_i$ countable infinite?
6. If $X_i$ is a countable infinite set, for $i = 1, \ldots, n$, is $\bigcup_i X_i$ countable infinite?
7. Let $X$ be a countable infinite set, and consider its power set

$$2^X = \{Y \mid Y \subseteq x\}.$$ 

The statement “the set $2^X$ is countable” is correct?
THE END

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(for now)