Describe recursive backtracking algorithms for the following problems. Don’t worry about running times.

1. Given an array $A[1..n]$ of integers, compute the length of a longest increasing subsequence.

**Solution (#1 of $\infty$):** Add a sentinel value $A[0] = -\infty$. Let $LIS(i, j)$ denote the length of the longest increasing subsequence of $A[j..n]$ where every element is larger than $A[i]$. This function obeys the following recurrence:

$$LIS(i, j) = \begin{cases} 
0 & \text{if } j > n \\
LIS(i, j + 1) & \text{if } j \leq n \text{ and } A[i] \geq A[j] \\
\max \{LIS(i, j + 1), 1 + LIS(j, j + 1)\} & \text{otherwise}
\end{cases}$$

We need to compute $LIS(0, 1)$. ■

**Solution (#2 of $\infty$):** Add a sentinel value $A[n+1] = -\infty$. Let $LIS(i, j)$ denote the length of the longest increasing subsequence of $A[1..j]$ where every element is smaller than $A[j]$. This function obeys the following recurrence:

$$LIS(i, j) = \begin{cases} 
0 & \text{if } i < 1 \\
LIS(i - 1, j) & \text{if } i \geq 1 \text{ and } A[i] \geq A[j] \\
\max \{LIS(i - 1, j), 1 + LIS(i - 1, i)\} & \text{otherwise}
\end{cases}$$

We need to compute $LIS(n, n + 1)$. ■

**Solution (#3 of $\infty$):** Let $LIS(i)$ denote the length of the longest increasing subsequence of $A[i..n]$ that begins with $A[i]$. This function obeys the following recurrence:

$$LIS(i) = \begin{cases} 
1 & \text{if } A[j] \leq A[i] \text{ for all } j > i \\
1 + \max \{LIS(j) \mid j > i \text{ and } A[j] > A[i]\} & \text{otherwise}
\end{cases}$$

(The first case is actually redundant if we define $\max \emptyset = 0$.) We need to compute $\max_i LIS(i)$. ■

**Solution (#4 of $\infty$):** Add a sentinel value $A[0] = -\infty$. Let $LIS(i)$ denote the length of the longest increasing subsequence of $A[i..n]$ that begins with $A[i]$. This function obeys the following recurrence:

$$LIS(i) = \begin{cases} 
1 & \text{if } A[j] \leq A[i] \text{ for all } j > i \\
1 + \max \{LIS(j) \mid j > i \text{ and } A[j] > A[i]\} & \text{otherwise}
\end{cases}$$

(The first case is actually redundant if we define $\max \emptyset = 0$.) We need to compute $LIS(0) - 1$; the $-1$ removes the sentinel $-\infty$ from the start of the subsequence. ■
**Solution (#5 of ∞):** Add sentinel values $A[0] = -\infty$ and $A[n + 1] = \infty$. Let $LIS(j)$ denote the length of the longest increasing subsequence of $A[1..j]$ that ends with $A[j]$. This function obeys the following recurrence:

$$LIS(j) = \begin{cases} 
1 & \text{if } j = 0 \\
1 + \max \{LIS(i) \mid i < j \text{ and } A[i] < A[j]\} & \text{otherwise}
\end{cases}$$

We need to compute $LIS(n + 1) - 2$; the $-2$ removes the sentinels $-\infty$ and $\infty$ from the subsequence.
2. Given an array \( A[1..n] \) of integers, compute the length of a *longest decreasing subsequence*.

**Solution (one of many):** Add a sentinel value \( A[0] = \infty \). Let \( LDS(i, j) \) denote the length of the longest decreasing subsequence of \( A[j..n] \) where every element is smaller than \( A[i] \). This function obeys the following recurrence:

\[
LDS(i, j) = \begin{cases} 
0 & \text{if } j > n \\
LDS(i, j + 1) & \text{if } j \leq n \text{ and } A[i] \leq A[j] \\
\max\{LDS(i, j + 1), 1 + LIS(j, j + 1)\} & \text{otherwise}
\end{cases}
\]

We need to compute \( LDS(0, 1) \).

**Solution (clever):** Multiply every element of \( A \) by \(-1\), and then compute the length of the longest increasing subsequence using the algorithm from problem 1.
Given an array $A[1..n]$ of integers, compute the length of a longest alternating subsequence.

**Solution (one of many):** We define two functions:

- Let $LAS^+(i, j)$ denote the length of the longest alternating subsequence of $A[j..n]$ whose first element (if any) is larger than $A[i]$ and whose second element (if any) is smaller than its first.
- Let $LAS^-(i, j)$ denote the length of the longest alternating subsequence of $A[j..n]$ whose first element (if any) is smaller than $A[i]$ and whose second element (if any) is larger than its first.

These two functions satisfy the following mutual recurrences:

$$
LAS^+(i, j) = \begin{cases} 
0 & \text{if } j > n \\
LAS^+(i, j + 1) & \text{if } j \leq n \text{ and } A[j] \leq A[i] \\
\max \{LAS^+(i, j + 1), 1 + LAS^-(j, j + 1)\} & \text{otherwise}
\end{cases}
$$

$$
LAS^-(i, j) = \begin{cases} 
0 & \text{if } j > n \\
LAS^-(i, j + 1) & \text{if } j \leq n \text{ and } A[j] \geq A[i] \\
\max \{LAS^-(i, j + 1), 1 + LAS^+(j, j + 1)\} & \text{otherwise}
\end{cases}
$$

To simplify computation, we consider two different sentinel values $A[0]$. First we set $A[0] = -\infty$ and let $\ell^+ = LAS^+(0, 1)$. Then we set $A[0] = +\infty$ and let $\ell^- = LAS^-(0, 1)$. Finally, the length of the longest alternating subsequence of $A$ is $\max\{\ell^+, \ell^-\}$. ■

**Solution (one of many):** We define two functions:

- Let $LAS^+(i)$ denote the length of the longest alternating subsequence of $A[i..n]$ that starts with $A[i]$ and whose second element (if any) is larger than $A[i]$.
- Let $LAS^-(i)$ denote the length of the longest alternating subsequence of $A[i..n]$ that starts with $A[i]$ and whose second element (if any) is smaller than $A[i]$.

These two functions satisfy the following mutual recurrences:

$$
LAS^+(i) = \begin{cases} 
1 & \text{if } A[j] \leq A[i] \text{ for all } j > i \\
1 + \max \{LAS^-(j) \mid j > i \text{ and } A[j] > A[i]\} & \text{otherwise}
\end{cases}
$$

$$
LAS^-(i) = \begin{cases} 
1 & \text{if } A[j] \geq A[i] \text{ for all } j > i \\
1 + \max \{LAS^+(j) \mid j > i \text{ and } A[j] < A[i]\} & \text{otherwise}
\end{cases}
$$

We need to compute $\max_i \max \{LAS^+(i), LAS^-(i)\}$. ■
To think about later:

4. Given an array \( A[1..n] \) of integers, compute the length of a longest convex subsequence of \( A \).

**Solution:** Let \( LCS(i, j) \) denote the length of the longest convex subsequence of \( A[i..n] \) whose first two elements are \( A[i] \) and \( A[j] \). This function obeys the following recurrence:

\[
LCS(i, j) = 1 + \max \{LCS(j, k) \mid j < k \leq n \text{ and } A[i] + A[k] > 2A[j]\}
\]

Here we define \( \max \emptyset = 0 \); this gives us a working base case. The length of the longest convex subsequence is \( \max_{1 \leq i < j \leq n} LCS(i, j) \).

**Solution (with sentinels):** Assume without loss of generality that \( A[i] \geq 0 \) for all \( i \). (Otherwise, we can add \( |m| \) to each \( A[i] \), where \( m \) is the smallest element of \( A[1..n] \).)

Add two sentinel values \( A[0] = 2M + 1 \) and \( A[-1] = 4M + 3 \), where \( M \) is the largest element of \( A[1..n] \).

Let \( LCS(i, j) \) denote the length of the longest convex subsequence of \( A[i..n] \) whose first two elements are \( A[i] \) and \( A[j] \). This function obeys the following recurrence:

\[
LCS(i, j) = 1 + \max \{LCS(j, k) \mid j < k \leq n \text{ and } A[i] + A[k] > 2A[j]\}
\]

Here we define \( \max \emptyset = 0 \); this gives us a working base case.

Finally, we claim that the length of the longest convex subsequence of \( A[1..n] \) is \( LCS(-1, 0) - 2 \).

**Proof:** First, consider any convex subsequence \( S \) of \( A[1..n] \), and suppose its first element is \( A[i] \). Then we have \( A[-1] - 2A[0] + A[i] = 4M + 3 - 2(2M + 1) + A[i] = A[i] + 1 > 0 \), which implies that \( A[-1] \cdot A[0] \cdot S \) is a convex subsequence of \( A[-1..n] \). So the longest convex subsequence of \( A[1..n] \) has length at most \( LCS(-1, 0) - 2 \).

On the other hand, removing \( A[-1] \) and \( A[0] \) from any convex subsequence of \( A[-1..n] \) leaves a convex subsequence of \( A[1..n] \). So the longest subsequence of \( A[1..n] \) has length at least \( LCS(-1, 0) - 2 \).
5. Given an array \( A[1..n] \), compute the length of a longest \textit{palindrome} subsequence of \( A \).

\textbf{Solution (naïve):} Let \( LPS(i, j) \) denote the length of the longest palindrome subsequence of \( A[i..j] \). This function obeys the following recurrence:

\[
LPS(i, j) = \begin{cases} 
0 & \text{if } i > j \\
1 & \text{if } i = j \\
\max \{ LPS(i + 1, j), LPS(i, j - 1) \} & \text{if } i < j \text{ and } A[i] \neq A[j] \\
2 + \max \{ LPS(i + 1, j - 1), LPS(i, j - 1) \} & \text{otherwise}
\end{cases}
\]

We need to compute \( LPS(1, n) \).

\textbf{Solution (with greedy optimization):} Let \( LPS(i, j) \) denote the length of the longest palindrome subsequence of \( A[i..j] \). Before stating a recurrence for this function, we make the following useful observation.\(^1\)

\textbf{Claim 1.} \textit{If } \( i < j \text{ and } A[i] = A[j] \text{, then } LPS(i, j) = 2 \text{ + } LPS(i + 1, j - 1) \).

\textbf{Proof:} Suppose \( i < j \) and \( A[i] = A[j] \). Fix an arbitrary longest palindrome subsequence \( S \) of \( A[i..j] \). There are four cases to consider.

- If \( S \) uses neither \( A[i] \) nor \( A[j] \), then \( A[i] \cdot S \cdot A[j] \) is a palindrome subsequence of \( A[i..j] \) that is longer than \( S \), which is impossible.
- Suppose \( S \) uses \( A[i] \) but not \( A[j] \). Let \( A[k] \) be the last element of \( S \). If \( k = i \), then \( A[i] \cdot A[k] \) is a palindrome subsequence of \( A[i..j] \) that is longer than \( S \), which is impossible. Otherwise, replacing \( A[k] \) with \( A[j] \) gives us a palindrome subsequence of \( A[i..j] \) with the same length as \( S \) that uses both \( A[i] \) and \( A[j] \).
- Suppose \( S \) uses \( A[j] \) but not \( A[i] \). Let \( A[h] \) be the first element of \( S \). If \( h = j \), then \( A[h] \cdot A[j] \) is a palindrome subsequence of \( A[i..j] \) that is longer than \( S \), which is impossible. Otherwise, replacing \( A[h] \) with \( A[i] \) gives us a palindrome subsequence of \( A[i..j] \) with the same length as \( S \) that uses both \( A[i] \) and \( A[j] \).
- Finally, \( S \) might include both \( A[i] \) and \( A[j] \).

In all cases, we find either a contradiction or a longest palindrome subsequence of \( A[i..j] \) that uses both \( A[i] \) and \( A[j] \).

Claim 1 implies that the function \( LPS \) satisfies the following recurrence:

\[
LPS(i, j) = \begin{cases} 
0 & \text{if } i > j \\
1 & \text{if } i = j \\
\max \{ LPS(i + 1, j), LPS(i, j - 1) \} & \text{if } i < j \text{ and } A[i] \neq A[j] \\
2 + \max \{ LPS(i + 1, j - 1), LPS(i, j - 1) \} & \text{otherwise}
\end{cases}
\]

We need to compute \( LPS(1, n) \).

\( ^1 \)And yes, optimizations like this require a proof of correctness, both in homework and on exams. Premature optimization is the root of all evil.