Let $L$ be an arbitrary regular language over the alphabet $\Sigma = \{0, 1\}$. Prove that the following languages are also regular. (You probably won’t get to all of these.)

1. **FlipOdds**($L$) := $\{\text{flipOdds}(w) \mid w \in L\}$, where the function $\text{flipOdds}$ inverts every odd-indexed bit in $w$. For example:

$$\text{flipOdds}(000111101010101) = 1010010111111111$$

**Solution:** Let $M = (Q, s, A, \delta)$ be a DFA that accepts $L$. We construct a new DFA $M' = (Q', s', A', \delta')$ that accepts $\text{FlipOdds}(L)$ as follows.

Intuitively, $M'$ receives some string $\text{flipOdds}(w)$ as input, restores every other bit to obtain $w$, and simulates $M$ on the restored string $w$.

Each state $(q, \text{flip})$ of $M'$ indicates that $M$ is in state $q$, and we need to flip the next input bit if $\text{flip} = \text{True}$.

$$Q' = Q \times \{\text{True}, \text{False}\}$$

$$s' = (s, \text{True})$$

$$A' = A \times \{\text{True}, \text{False}\}$$

$$\delta'((q, \text{flip}), a) = (\delta(q, a \oplus \text{flip}), \neg \text{flip})$$

Here I am treating 1 and 0 as synonyms for True and False, respectively. ■

2. **UnflipOdd1s**($L$) := $\{w \in \Sigma^* \mid \text{flipOdd1s}(w) \in L\}$, where the function $\text{flipOdd1s}$ inverts every other 1 bit of its input string, starting with the first 1. For example:

$$\text{flipOdd1s}(000111101010101) = 0000101001001001$$

**Solution:** Let $M = (Q, s, A, \delta)$ be a DFA that accepts $L$. We construct a new DFA $M' = (Q', s', A', \delta')$ that accepts $\text{UnflipOdd1s}(L)$ as follows.

Intuitively, $M'$ receives some string $w$ as input, flips every other 1 bit, and simulates $M$ on the transformed string.

Each state $(q, \text{flip})$ of $M'$ indicates that $M$ is in state $q$, and we need to flip the next 1 bit of and only if $\text{flip} = \text{True}$.

$$Q' = Q \times \{\text{True}, \text{False}\}$$

$$s' = (s, \text{True})$$

$$A' = A \times \{\text{True}, \text{False}\}$$

$$\delta'((q, \text{flip}), a) = (\delta(q, flip \oplus a), flip \oplus a)$$

Again, I am treating 1 and 0 as synonyms for True and False, respectively. ■
3. \( \text{FlipOdd}^1(L) := \{ \text{flipOdd}^1(w) \mid w \in L \} \), where the function \( \text{flipOdd}^1 \) is defined as in the previous problem.

**Solution:** Let \( M = (Q,s,A,\delta) \) be a DFA that accepts \( L \). We construct a new NFA \( M' = (Q',s',A',\delta') \) that accepts \( \text{FlipOdd}^1(L) \) as follows.

Intuitively, \( M' \) receives some string \( \text{flipOdd}^1(w) \) as input, guesses which 0 bits to restore to 1s, and simulates \( M \) on the restored string \( w \). No string in \( \text{FlipOdd}^1(L) \) has two 1s in a row, so if \( M' \) ever sees 11, it rejects.

Each state \((q,\text{flip})\) of \( M' \) indicates that \( M \) is in state \( q \), and we need to flip a 0 bit before the next 1 bit if and only if \( \text{flip} = \text{True} \).

\[
\begin{align*}
Q' &= Q \times \{ \text{True}, \text{False} \} \\
q' &= (s, \text{True}) \\
A' &= A \times \{ \text{True}, \text{False} \} \\
\delta'((q,\text{False}),\theta) &= \{ (\delta(q,\theta), \text{False}) \} \\
\delta'((q,\text{True}),\theta) &= \{ (\delta(q,\theta), \text{True}, (\delta(q,1), \text{False}) \} \\
\delta'((q,\text{False}),1) &= \{ (\delta(q,1), \text{True}) \} \\
\delta'((q,\text{True}),1) &= \emptyset
\end{align*}
\]

The last transition indicates that we waited too long to flip a 0 to a 1, so we should kill the current execution thread. ■

4. \( \text{Faro}(L) := \{ \text{faro}(w,x) \mid w,x \in L \text{ and } |w| = |x| \} \), where the function \( \text{faro} \) is defined recursively as follows:

\[
\text{faro}(w,x) := \begin{cases} 
x & \text{if } w = \epsilon \\
a \cdot \text{faro}(x,y) & \text{if } w = ax \text{ for some } a \in \Sigma \text{ and some } y \in \Sigma^* \end{cases}
\]

**Solution:** Let \( M = (Q,s,A,\delta) \) be a DFA that accepts \( L \). We construct a DFA \( M' = (Q',s',A',\delta') \) that accepts \( \text{Faro}(L) \) as follows.

Intuitively, \( M' \) reads the string \( \text{faro}(w,x) \) as input, splits the string into the subsequences \( w \) and \( x \), and passes each of those strings to an independent copy of \( M \).

Each state \((q_1,q_2,\text{next})\) indicates that the copy of \( M \) that gets \( w \) is in state \( q_1 \), the copy of \( M \) that gets \( x \) is in state \( q_2 \), and \( \text{next} \) indicates which copy gets the next input bit. Because of the constraint \( |w| = |x| \), machine \( M' \) can accept only if \( \text{next} = 1 \).

\[
\begin{align*}
Q' &= Q \times Q \times \{ 1,2 \} \\
q' &= (s,s,1) \\
A' &= \{ (q_1,q_2,1) \mid q_1,q_2 \in A \} \\
\delta'((q_1,q_2,\text{next}),a) &= \begin{cases} 
(\delta(q_1,a),q_2,2) & \text{if } \text{next} = 1 \\
(q_1,\delta(q_2,a),1) & \text{if } \text{next} = 2
\end{cases}
\end{align*}
\]

■