Prove that each of the following languages is not regular.

1. \( \{ \theta^{2^n} \mid n \geq 0 \} \)

**Solution (verbose):** Let \( F = L = \{ \theta^{2^n} \mid n \geq 0 \} \).

Let \( x \) and \( y \) be arbitrary elements of \( F \).

Then \( x = \theta^{2^i} \) and \( y = \theta^{2^j} \) for some non-negative integers \( x \) and \( y \).

Let \( z = \theta^{2^i} \).

Then \( xz = \theta^{2^i} \theta^{2^i} = \theta^{2^{i+1}} \in L \).

And \( yz = \theta^{2^i} \theta^{2^j} = \theta^{2^i+2^j} \notin L \), because \( i \neq j \)

Thus, \( F \) is a fooling set for \( L \).

Because \( F \) is infinite, \( L \) cannot be regular.

**Solution (concise):** For any non-negative integers \( i \neq j \), the strings \( \theta^{2^i} \) and \( \theta^{2^j} \) are distinguished by the suffix \( \theta^{2^i} \), because \( \theta^{2^i} \theta^{2^i} = \theta^{2^{i+1}} \in L \) but \( \theta^{2^i} \theta^{2^j} = \theta^{2^{i+j}} \notin L \). Thus \( L \) itself is an infinite fooling set for \( L \). ■

2. \( \{ \theta^{2n}1^n \mid n \geq 0 \} \)

**Solution (verbose):** Let \( F \) be the language \( \theta^* \).

Let \( x \) and \( y \) be arbitrary strings in \( F \).

Then \( x = \theta^i \) and \( y = \theta^j \) for some non-negative integers \( i \neq j \).

Let \( z = \theta^i1^i \).

Then \( xz = \theta^{2i}1^i \in L \).

And \( yz = \theta^{i+j}1^i \notin L \), because \( i + j \neq 2i \).

Thus, \( F \) is a fooling set for \( L \).

Because \( F \) is infinite, \( L \) cannot be regular.

**Solution (concise):** For all non-negative integers \( i \neq j \), the strings \( \theta^i \) and \( \theta^j \) are distinguished by the suffix \( \theta^i1^i \), because \( \theta^{2i}1^i \in L \) but \( \theta^{i+j}1^i \notin L \). Thus, the language \( \theta^* \) is an infinite fooling set for \( L \). ■

**Solution (concise, different fooling set):** For all non-negative integers \( i \neq j \), the strings \( \theta^{2i} \) and \( \theta^{2j} \) are distinguished by the suffix \( 1^i \), because \( \theta^{2i}1^i \in L \) but \( \theta^{2j}1^i \notin L \). Thus, the language \( (\theta0)^* \) is an infinite fooling set for \( L \). ■
3. \{0^m1^n \mid m \neq 2n\}

**Solution (verbose):** Let \(F\) be the language \(0^*\).

Let \(x\) and \(y\) be arbitrary strings in \(F\).

Then \(x = 0^i\) and \(y = 0^j\) for some non-negative integers \(i \neq j\).

Let \(z = 0^i1^i\).

Then \(xz = 0^{2i}1^i \notin L\).

And \(yz = 0^{i+j}1^i \in L\), because \(i + j \neq 2i\).

Thus, \(F\) is a fooling set for \(L\).

Because \(F\) is infinite, \(L\) cannot be regular.

**Solution (concise, different fooling set):** For all non-negative integers \(i \neq j\), the strings \(0^{2i}\) and \(0^{2j}\) are distinguished by the suffix \(1^i\), because \(0^{2i}1^i \notin L\) but \(0^{2j}1^i \in L\). Thus, the language \((00)^*\) is an infinite fooling set for \(L\).

4. Strings over \(\{0, 1\}\) where the number of 0s is exactly twice the number of 1s.

**Solution (verbose):** Let \(F\) be the language \(0^*\).

Let \(x\) and \(y\) be arbitrary strings in \(F\).

Then \(x = 0^i\) and \(y = 0^j\) for some non-negative integers \(i \neq j\).

Let \(z = 0^i1^i\).

Then \(xz = 0^{2i}1^i \in L\).

And \(yz = 0^{i+j}1^i \notin L\), because \(i + j \neq 2i\).

Thus, \(F\) is a fooling set for \(L\).

Because \(F\) is infinite, \(L\) cannot be regular.

**Solution (concise, different fooling set):** For all non-negative integers \(i \neq j\), the strings \(0^{2i}\) and \(0^{2j}\) are distinguished by the suffix \(1^i\), because \(0^{2i}1^i \in L\) but \(0^{2j}1^i \notin L\). Thus, the language \((00)^*\) is an infinite fooling set for \(L\).

**Solution (closure properties):** If \(L\) were regular, then the language

\[
\left( (0 + 1)^* \setminus L \right) \cap 0^*1^* = \{0^m1^n \mid m \neq 2n\}
\]

would also be regular, because regular languages are closed under complement and intersection. But we just proved that \(\{0^m1^n \mid m \neq 2n\}\) is not regular in problem 3. [Yes, this proof would be worth full credit, either in homework or on an exam.]
5. Strings of properly nested parentheses ( ), brackets [], and braces {}. For example, the string ( [ ] ) {} is in this language, but the string ( [ ] ) is not, because the left and right delimiters don’t match.

**Solution (verbose):** Let $F$ be the language ($^*$).

Let $x$ and $y$ be arbitrary strings in $F$.

Then $x = (^i$ and $y = (^j$ for some non-negative integers $i \neq j$.

Let $z = )^i$.

Then $xz = (^i)^i \in L$.

And $yz = (^j)^i \notin L$, because $i \neq j$.

Thus, $F$ is a fooling set for $L$.

Because $F$ is infinite, $L$ cannot be regular.

**Solution (concise):** For any non-negative integers $i \neq j$, the strings $^i$ and $^j$ are distinguished by the suffix $)^i$, because $^i)^i \in L$ but $^j)^i \notin L$. Thus, the language ($^*$ is an infinite fooling set.

6. Strings of the form $w_1\#w_2\#\cdots\#w_n$ for some $n \geq 2$, where each substring $w_i$ is a string in $\{0, 1\}^*$, and some pair of substrings $w_i$ and $w_j$ are equal.

**Solution (verbose):** Let $F$ be the language $\theta^*$.

Let $x$ and $y$ be arbitrary strings in $F$.

Then $x = \theta^i$ and $y = \theta^j$ for some non-negative integers $i \neq j$.

Let $z = \#\theta^i$.

Then $xz = \theta^i\#\theta^i \in L$.

And $yz = \theta^j\#\theta^i \notin L$, because $i \neq j$.

Thus, $F$ is a fooling set for $L$.

Because $F$ is infinite, $L$ cannot be regular.

**Solution (concise):** For any non-negative integers $i \neq j$, the strings $\theta^i$ and $\theta^j$ are distinguished by the suffix $\#\theta^i$, because $\theta^i\#\theta^i \in L$ but $\theta^i\#\theta^j \notin L$. Thus, the language $\theta^*$ is an infinite fooling set.
Work on these later:

7. \( \{ \sigma^{n^2} \mid n \geq 0 \} \)

**Solution:** Let \( x \) and \( y \) be distinct arbitrary strings in \( L \).

Without loss of generality, \( x = \sigma^{i^2} \) and \( y = \sigma^{j^2} \) for some \( i > j \geq 0 \).

Let \( z = \sigma^{2i+1} \).

Then \( xz = \sigma^{i^2+2i+1} = \sigma^{(i+1)^2} \in L \).

On the other hand, \( yz = \sigma^{i^2+2j+1} \notin L \), because \( i^2 < i^2 + 2j + 1 < (i + 1)^2 \).

Thus, \( z \) distinguishes \( x \) and \( y \).

We conclude that \( L \) is an infinite fooling set for \( L \), so \( L \) cannot be regular. ■

**Solution:** Let \( x \) and \( y \) be distinct arbitrary strings in \( \sigma^* \).

Without loss of generality, \( x = \sigma^i \) and \( y = \sigma^j \) for some \( i > j \geq 0 \).

Let \( z = \sigma^{i^2+i+1} \).

Then \( xz = \sigma^{i^2+2i+1} = \sigma^{(i+1)^2} \in L \).

On the other hand, \( yz = \sigma^{i^2+i+j+1} \notin L \), because \( i^2 < i^2 + i + j + 1 < (i + 1)^2 \).

Thus, \( z \) distinguishes \( x \) and \( y \).

We conclude that \( \sigma^* \) is an infinite fooling set for \( L \), so \( L \) cannot be regular. ■

**Solution:** Let \( x \) and \( y \) be distinct arbitrary strings in \( \sigma 0000^* \).

Without loss of generality, \( x = \sigma^i \) and \( y = \sigma^j \) for some \( i > j \geq 3 \).

Let \( z = \sigma^{i^2-i} \).

Then \( xz = \sigma^{i^2} \in L \).

On the other hand, \( yz = \sigma^{i^2-i+j} \notin L \), because

\[
(i - 1)^2 = i^2 - 2i + 1 < i^2 - i < i^2 - i + j < i^2.
\]

(The first inequalities requires \( i \geq 2 \), and the second \( j \geq 1 \).)

Thus, \( z \) distinguishes \( x \) and \( y \).

We conclude that \( \sigma 0000^* \) is an infinite fooling set for \( L \), so \( L \) cannot be regular. ■
8. \{w \in (0+1)^* \mid w \text{ is the binary representation of a perfect square}\}

**Solution:** We design our fooling set around numbers of the form \((2^k+1)^2 = 2^{2k} + 2^{k+1} + 1 = 10^{k-2}10^k 1 \in L\), for any integer \(k \geq 2\). The argument is somewhat simpler if we further restrict \(k\) to be even.

Let \(F = 1(00)^*1\), and let \(x\) and \(y\) be arbitrary strings in \(F\).

Then \(x = 10^{2i-2}1\) and \(y = 10^{2j-2}1\), for some positive integers \(i \neq j\).

Without loss of generality, assume \(i < j\). (Otherwise, swap \(x\) and \(y\).)

Let \(z = 0^{2i}1\).

Then \(xz = 10^{2i}10^{2j}1\) is the binary representation of \(2^{4i} + 2^{2i+1} + 1 = (2^{2i} + 1)^2\), and therefore \(xz \in L\).

On the other hand, \(yz = 10^{2j-2}10^{2i}1\) is the binary representation of \(2^{2i} + 2^{2j+1} + 1\). Simple algebra gives us the inequalities

\[
(2^{i+j})^2 = 2^{2i+2j} < 2^{2i+2j} + 2^{2i+1} + 1 < 2^{2(i+j)} + 2^{i+j+1} + 1 = (2^{i+j} + 1)^2.
\]

So \(2^{2i+2j} + 2^{2i+1} + 1\) lies between two consecutive perfect squares, and thus is not a perfect square, which implies that \(yz \notin L\).

We conclude that \(F\) is a fooling set for \(L\). Because \(F\) is infinite, \(L\) cannot be regular. ■