Proving that a problem \( X \) is NP-hard requires several steps:

- Choose a problem \( Y \) that you already know is NP-hard (because we told you so in class).
- Describe an algorithm to solve \( Y \), using an algorithm for \( X \) as a subroutine. Typically this algorithm has the following form: Given an instance of \( Y \), transform it into an instance of \( X \), and then call the magic black-box algorithm for \( X \).
- **Prove** that your algorithm is correct. This always requires two separate steps, which are usually of the following form:
  - **Prove** that your algorithm transforms “good” instances of \( Y \) into “good” instances of \( X \).
  - **Prove** that your algorithm transforms “bad” instances of \( Y \) into “bad” instances of \( X \). Equivalently: Prove that if your transformation produces a “good” instance of \( X \), then it was given a “good” instance of \( Y \).
- Argue that your algorithm for \( Y \) runs in polynomial time.

1. Recall the following \( k\text{COLOR} \) problem: Given an undirected graph \( G \), can its vertices be colored with \( k \) colors, so that every edge touches vertices with two different colors?
   
   (a) Describe a direct polynomial-time reduction from \( 3\text{COLOR} \) to \( 4\text{COLOR} \).
   (b) Prove that \( k\text{COLOR} \) problem is NP-hard for any \( k \geq 3 \).

2. A Hamiltonian cycle in a graph \( G \) is a cycle that goes through every vertex of \( G \) exactly once. Deciding whether an arbitrary graph contains a Hamiltonian cycle is NP-hard.

   A tonian cycle in a graph \( G \) is a cycle that goes through at least half of the vertices of \( G \). Prove that deciding whether a graph contains a tonian cycle is NP-hard.

To think about later:

3. Let \( G \) be an undirected graph with weighted edges. A Hamiltonian cycle in \( G \) is **heavy** if the total weight of edges in the cycle is at least half of the total weight of all edges in \( G \). Prove that deciding whether a graph contains a heavy Hamiltonian cycle is NP-hard.

A heavy Hamiltonian cycle. The cycle has total weight 34; the graph has total weight 67.