Consider the following recursively defined function on strings:

\[
\text{stutter}(w) := \begin{cases} \\
\epsilon & \text{if } w = \epsilon \\
aa \cdot \text{stutter}(x) & \text{if } w = ax \text{ for some symbol } a \text{ and some string } x
\end{cases}
\]

Intuitively, \(\text{stutter}(w)\) doubles every symbol in \(w\). For example:

- \(\text{stutter}(\text{PRESTO}) = \text{PPRREESSTTOO}\)
- \(\text{stutter}(\text{HOCUS}\diamond\text{POCUS}) = \text{HHOOCCUUSS}\diamond\diamond\text{PPOOCCUUSS}\)

Let \(L\) be an arbitrary regular language.

1. Prove that the language \(\text{stutter}^{-1}(L) := \{w \mid \text{stutter}(w) \in L\}\) is regular.
2. Prove that the language \(\text{stutter}(L) := \{\text{stutter}(w) \mid w \in L\}\) is regular.

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Work on these later:

3. Let \(L\) be an arbitrary regular language.

   (a) Prove that the language \(\text{insert}^1(L) := \{x1y \mid xy \in L\}\) is regular.

   Intuitively, \(\text{insert}^1(L)\) is the set of all strings that can be obtained from strings in \(L\) by inserting exactly one \(1\). For example, if \(L = \{\epsilon, \text{OOK!}\}\), then \(\text{insert}^1(L) = \{1, \text{1OOK!}, \text{01OK!}, \text{001K!}, \text{00K!1}, \text{OOK!1}\}\).

   (b) Prove that the language \(\text{delete}^1(L) := \{xy \mid x1y \in L\}\) is regular.

   Intuitively, \(\text{delete}^1(L)\) is the set of all strings that can be obtained from strings in \(L\) by deleting exactly one \(1\). For example, if \(L = \{101101, 00, \epsilon\}\), then \(\text{delete}^1(L) = \{01101, 10101, 10110\}\).

4. Consider the following recursively defined function on strings:

\[
\text{evens}(w) := \begin{cases} \\
\epsilon & \text{if } w = \epsilon \\
\epsilon & \text{if } w = a \text{ for some symbol } a \\
b \cdot \text{evens}(x) & \text{if } w = abx \text{ for some symbols } a \text{ and } b \text{ and some string } x
\end{cases}
\]

Intuitively, \(\text{evens}(w)\) skips over every other symbol in \(w\). For example:

- \(\text{evens}(\text{EXPELLIARMUS}) = \text{XELAMS}\)
- \(\text{evens}(\text{AVADA}\diamond\text{KEDAVRA}) = \text{VD}\diamond\text{EAR}\).

Once again, let \(L\) be an arbitrary regular language.

   (a) Prove that the language \(\text{evens}^{-1}(L) := \{w \mid \text{evens}(w) \in L\}\) is regular.
   (b) Prove that the language \(\text{evens}(L) := \{\text{evens}(w) \mid w \in L\}\) is regular.