Even More on Dynamic Programming

Lecture 15
Thursday, October 19, 2017
Part I

Longest Common Subsequence Problem
The LCS Problem

**Definition**

LCS between two strings $X$ and $Y$ is the length of longest common subsequence between $X$ and $Y$.

**Example**

LCS between ABAZDC and BACBAD is 4 via ABAD

Derive a dynamic programming algorithm for the problem.
**The LCS Problem**

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The LCS Problem

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\text{LCS} between two strings \textbf{X} and \textbf{Y} is the length of longest common subsequence between \textbf{X} and \textbf{Y}.

Example

\text{LCS} between ABAZDC and BACBAD is 4 via ABAD

Derive a dynamic programming algorithm for the problem.
Part II

Maximum Weighted Independent Set in Trees
Maximum Weight Independent Set Problem

Input  Graph $G = (V, E)$ and weights $w(v) \geq 0$ for each $v \in V$

Goal  Find maximum weight independent set in $G$

Maximum weight independent set in above graph: $\{B, D\}$
Maximum Weight Independent Set Problem

Input  Graph $G = (V, E)$ and weights $w(v) \geq 0$ for each $v \in V$

Goal  Find maximum weight independent set in $G$

Maximum weight independent set in above graph: $\{B, D\}$
Input  Tree \( T = (V, E) \) and weights \( w(v) \geq 0 \) for each \( v \in V \)

Goal  Find maximum weight independent set in \( T \)

Maximum weight independent set in above tree: ??
Towards a Recursive Solution

For an arbitrary graph $G$:

1. Number vertices as $v_1, v_2, \ldots, v_n$
2. Find recursively optimum solutions without $v_n$ (recurse on $G - v_n$) and with $v_n$ (recurse on $G - v_n - N(v_n)$ & include $v_n$).
3. Saw that if graph $G$ is arbitrary there was no good ordering that resulted in a small number of subproblems.

What about a tree? Natural candidate for $v_n$ is root $r$ of $T$?
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What about a tree? Natural candidate for $v_n$ is root $r$ of $T$?
Towards a Recursive Solution

Natural candidate for $v_n$ is root $r$ of $T$? Let $O$ be an optimum solution to the whole problem.

Case $r \not\in O$ : Then $O$ contains an optimum solution for each subtree of $T$ hanging at a child of $r$.

Case $r \in O$ : None of the children of $r$ can be in $O$. $O - \{r\}$ contains an optimum solution for each subtree of $T$ hanging at a grandchild of $r$.

Subproblems? Subtrees of $T$ rooted at nodes in $T$.

How many of them? $O(n)$
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How many of them? $O(n)$
Example
A Recursive Solution

\( T(u) \): subtree of \( T \) hanging at node \( u \)

\( OPT(u) \): max weighted independent set value in \( T(u) \)

\[
OPT(u) = \max \left\{ \sum_{v \text{ child of } u} OPT(v), \quad w(u) + \sum_{v \text{ grandchild of } u} OPT(v) \right\}
\]
A Recursive Solution

$T(u)$: subtree of $T$ hanging at node $u$

$OPT(u)$: max weighted independent set value in $T(u)$

$$OPT(u) = \max \left\{ \sum_{v \text{ child of } u} OPT(v), \right.$$

$$w(u) + \sum_{v \text{ grandchild of } u} OPT(v) \right\}$$
Iterative Algorithm

1. Compute $OPT(u)$ bottom up. To evaluate $OPT(u)$ need to have computed values of all children and grandchildren of $u$.

2. What is an ordering of nodes of a tree $T$ to achieve above?
   
   Post-order traversal of a tree.
Iterative Algorithm

1. Compute $\text{OPT}(u)$ bottom up. To evaluate $\text{OPT}(u)$ need to have computed values of all children and grandchildren of $u$.

2. What is an ordering of nodes of a tree $T$ to achieve above? Post-order traversal of a tree.
Iterative Algorithm

**MIS-Tree** ($T$):

Let $v_1, v_2, \ldots, v_n$ be a post-order traversal of nodes of $T$

for $i = 1$ to $n$ do

\[
M[v_i] = \max \left( \sum_{j \text{ child of } v_i} M[v_j], \quad w(v_i) + \sum_{j \text{ grandchild of } v_i} M[v_j] \right)
\]

return $M[v_n]$ (* Note: $v_n$ is the root of $T$ *)

**Space**: $O(n)$ to store the value at each node of $T$

**Running time:**

1. Naive bound: $O(n^2)$ since each $M[v_i]$ evaluation may take $O(n)$ time and there are $n$ evaluations.

2. Better bound: $O(n)$. A value $M[v_j]$ is accessed only by its parent and grand parent.
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w($v_i$) + $w(v_i)$

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Iterative Algorithm

\textbf{MIS-Tree}(T):

Let $v_1, v_2, \ldots, v_n$ be a post-order traversal of nodes of T

\begin{algorithm}
\begin{algorithmic}
  \For{$i = 1$ to $n$}
    \State $M[v_i] = \max \left( \sum_{v_j \text{ child of } v_i} M[v_j], w(v_i) + \sum_{v_j \text{ grandchild of } v_i} M[v_j] \right)$
  \EndFor
\end{algorithmic}
\end{algorithm}

\Return $M[v_n]$ (* Note: $v_n$ is the root of T *)

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1. Naive bound: $O(n^2)$ since each $M[v_i]$ evaluation may take $O(n)$ time and there are $n$ evaluations.

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Example

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```
Part III

Context free grammars: The CYK Algorithm
We saw regular languages and context free languages.

Most programming languages are specified via context-free grammars. Why?

- **CFLs** are sufficiently expressive to support what is needed.
- At the same time one can “efficiently” solve the parsing problem: given a string/program $w$, is it a valid program according to the CFG specification of the programming language?
CFG specification for C

\[
<\text{relational-expression}> ::= <\text{shift-expression}>
  
  | <\text{relational-expression}> < <\text{shift-expression}>
  
  | <\text{relational-expression}> > <\text{shift-expression}>
  
  | <\text{relational-expression}> <= <\text{shift-expression}>
  
  | <\text{relational-expression}> >= <\text{shift-expression}>

<\text{shift-expression}> ::= <\text{additive-expression}>
  
  | <\text{shift-expression}> << <\text{additive-expression}>
  
  | <\text{shift-expression}> >> <\text{additive-expression}>

<\text{additive-expression}> ::= <\text{multiplicative-expression}>
  
  | <\text{additive-expression}> + <\text{multiplicative-expression}>
  
  | <\text{additive-expression}> - <\text{multiplicative-expression}>

<\text{multiplicative-expression}> ::= <\text{cast-expression}>
  
  | <\text{multiplicative-expression}> * <\text{cast-expression}>
  
  | <\text{multiplicative-expression}> / <\text{cast-expression}>
  
  | <\text{multiplicative-expression}> % <\text{cast-expression}>

<\text{cast-expression}> ::= <\text{unary-expression}>
  
  | ( <\text{type-name}>) <\text{cast-expression}>

<\text{unary-expression}> ::= <\text{postfix-expression}>
  
  | ++ <\text{unary-expression}>
  
  | -- <\text{unary-expression}>
  
  | <\text{unary-operator}> <\text{cast-expression}>
  
  | \text{sizeof} <\text{unary-expression}>
  
  | \text{sizeof} <\text{type-name}>
Algorithmic Problem

Given a CFG \( G = (V, T, P, S) \) and a string \( w \in T^* \), is \( w \in L(G) \)?

- That is, does \( S \) derive \( w \)?
- Equivalently, is there a parse tree for \( w \)?

Simplifying assumption: \( G \) is in Chomsky Normal Form (CNF)

- Productions are all of the form \( A \rightarrow BC \) or \( A \rightarrow a \).
  - If \( \epsilon \in L \) then \( S \rightarrow \epsilon \) is also allowed.
    - (This is the only place in the grammar that has an \( \epsilon \).)
- Every CFG \( G \) can be converted into CNF form via an efficient algorithm.
- Advantage: parse tree of constant degree.
Algorithmic Problem

Given a $\text{CFG } G = (V, T, P, S)$ and a string $w \in T^*$, is $w \in L(G)$?

- That is, does $S$ derive $w$?
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- Every $\text{CFG } G$ can be converted into CNF form via an efficient algorithm.

- Advantage: parse tree of constant degree.
CYK Algorithm = Cocke–Younger–Kasami algorithm
Example

\[
S \to \epsilon \mid AB \mid XB \\
Y \to AB \mid XB \\
X \to AY \\
A \to 0 \\
B \to 1
\]

Question:
- Is 000111 in \( L(G) \)?
- Is 00011 in \( L(G) \)?
Towards Recursive Algorithm

Assume $G$ is a CNF grammar.

$S$ derives $w$ iff one of the following holds:

- $|w| = 1$ and $S \rightarrow w$ is a rule in $P$
- $|w| > 1$ and there is a rule $S \rightarrow AB$ and a split $w = uv$ with $|u|, |v| \geq 1$ such that $A$ derives $u$ and $B$ derives $v$

Observation: Subproblems generated require us to know if some non-terminal $A$ will derive a substring of $w$. 
Towards Recursive Algorithm

Assume $G$ is a CNF grammar. 
$S$ derives $w$ iff one of the following holds:

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Observation: Subproblems generated require us to know if some non-terminal $A$ will derive a substring of $w$. 
Recursive solution

1. Input: \( w = w_1 w_2 \ldots w_n \)
2. Assume \( r \) non-terminals in \( G: R_1, \ldots, R_r \).
3. \( R_1 \): Start symbol.
4. \( f(\ell, s, b) \): TRUE \( \iff w_s w_{s+1} \ldots, w_{s+\ell-1} \in L(R_b) \).

   = Substring \( w \) starting at pos \( \ell \) of length \( s \) is deriveable by \( R_b \).
5. Recursive formula: \( f(1, s, a) \) is 1 iff \( (R_a \rightarrow w_s) \in G \).
6. For \( \ell > 1 \):

   \[
   f(\ell, s, a) = \bigvee_{p=1}^{\ell-1} \bigvee_{(R_a \rightarrow R_b R_c) \in G} (f(p, s, b) \land f(\ell - p, s + p, c))
   \]
7. Output: \( w \in L(G) \iff f(n, 1, 1) = 1. \)
Recursive solution

1. Input: \( w = w_1 w_2 \ldots w_n \)
2. Assume \( r \) non-terminals in \( G: R_1, \ldots, R_r \).
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4. \( f(\ell, s, b) \): TRUE \iff \( w_s w_{s+1} \ldots, w_{s+\ell-1} \in L(R_b) \).
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f(\ell, s, a) = \bigvee_{p=1}^{\ell-1} \bigvee (R_a \rightarrow R_b R_c) \in G \quad (f(p, s, b) \land f(\ell - p, s + p, c))
\]

7. Output: \( w \in L(G) \iff f(n, 1, 1) = 1 \).
Assume $G = \{R_1, R_2, \ldots, R_r\}$ with start symbol $R_1$

- Number of subproblems: $O(rn^2)$
- Space: $O(rn^2)$
- Time to evaluate a subproblem from previous ones: $O(|P|n)$ where $P$ is set of rules
- Total time: $O(|P|rn^3)$ which is polynomial in both $|w|$ and $|G|$. For fixed $G$ the run time is cubic in input string length.
- Running time can be improved to $O(n^3|P|)$.
- Not practical for most programming languages. Most languages assume restricted forms of C that enable more efficient parsing algorithms.
Input string: $X = x_1 \ldots x_n$.
Input grammar $G$: $r$ nonterminal symbols $R_1 \ldots R_r$, $R_1$ start symbol.

$P[n][n][r]$: Array of booleans. Initialize all to FALSE

for $s = 1$ to $n$ do
    for each unit production $R_v \rightarrow x_s$ do
        $P[1][s][v] \leftarrow$ TRUE

for $\ell = 2$ to $n$ do // Length of span
    for $s = 1$ to $n - \ell + 1$ do // Start of span
        for $p = 1$ to $\ell - 1$ do // Partition of span
            for all $(R_a \rightarrow R_b R_c) \in G$ do
                if $P[p][s][b]$ and $P[l - p][s + p][c]$ then
                    $P[l][s][a] \leftarrow$ TRUE

if $P[n][1][1]$ is TRUE then
    return ""$X$ is member of language"
else
    return ""$X$ is not member of language""
Example

\[ S \rightarrow \epsilon | AB | XB \]
\[ Y \rightarrow AB | XB \]
\[ X \rightarrow AY \]
\[ A \rightarrow 0 \]
\[ B \rightarrow 1 \]

Question:
- Is \textbf{000111} in \( L(G) \)?
- Is \textbf{00011} in \( L(G) \)?

**Order of evaluation for iterative algorithm:** increasing order of substring length.
Example

\[
S \rightarrow \epsilon \mid AB \mid XB \\
Y \rightarrow AB \mid XB \\
X \rightarrow AY \\
A \rightarrow 0 \\
B \rightarrow 1
\]
Takeaway Points

1. Dynamic programming is based on finding a recursive way to solve the problem. Need a recursion that generates a small number of subproblems.

2. Given a recursive algorithm there is a natural DAG associated with the subproblems that are generated for given instance; this is the dependency graph. An iterative algorithm simply evaluates the subproblems in some topological sort of this DAG.

3. The space required to evaluate the answer can be reduced in some cases by a careful examination of that dependency DAG of the subproblems and keeping only a subset of the DAG at any time.