Backtracking and Memoization

Lecture 12
Tuesday, October 10, 2017
Recursion

Reduction:
Reduce one problem to another

Recursion
A special case of reduction
1. reduce problem to a smaller instance of itself
2. self-reduction

1. Problem instance of size $n$ is reduced to one or more instances of size $n - 1$ or less.
2. For termination, problem instances of small size are solved by some other method as base cases.
Recursion in Algorithm Design

1. **Tail Recursion**: problem reduced to a *single* recursive call after some work. Easy to convert algorithm into iterative or greedy algorithms. Examples: Interval scheduling, MST algorithms, etc.

2. **Divide and Conquer**: Problem reduced to multiple *independent* sub-problems that are solved separately. Conquer step puts together solution for bigger problem. Examples: Closest pair, deterministic median selection, quick sort.

3. **Backtracking**: Refinement of brute force search. Build solution incrementally by invoking recursion to try all possibilities for the decision in each step.

4. **Dynamic Programming**: problem reduced to multiple (typically) *dependent or overlapping* sub-problems. Use *memoization* to avoid recomputation of common solutions leading to *iterative bottom-up* algorithm.
Part 1

Brute Force Search, Recursion and Backtracking
Maximum Independent Set in a Graph

Definition

Given undirected graph $G = (V, E)$ a subset of nodes $S \subseteq V$ is an independent set (also called a stable set) if for there are no edges between nodes in $S$. That is, if $u, v \in S$ then $(u, v) \notin E$.

Some independent sets in graph above: \{D\}, \{A, C\}, \{B, E, F\}
Maximum Independent Set Problem

Input  Graph $G = (V, E)$

Goal  Find maximum sized independent set in $G$
Maximum Weight Independent Set Problem

Input  Graph $G = (V, E)$, weights $w(v) \geq 0$ for $v \in V$
Goal   Find maximum weight independent set in $G$
Maximum Weight Independent Set Problem

1. No one knows an *efficient* (polynomial time) algorithm for this problem.

2. Problem is **NP-Complete** and it is *believed* that there is no polynomial time algorithm.

**Brute-force algorithm:**

Try all subsets of vertices.
Algorithm to find the size of the maximum weight independent set.

\textbf{MaxIndSet} \((G = (V, E))\):
\begin{itemize}
  \item \(\text{max} = 0\)
  \item \textbf{for} each subset \(S \subseteq V\) \textbf{do}
    \begin{itemize}
      \item check if \(S\) is an independent set
      \item \textbf{if} \(S\) is an independent set and \(w(S) > \text{max}\) \textbf{then}
        \begin{itemize}
          \item \(\text{max} = w(S)\)
        \end{itemize}
    \end{itemize}
\end{itemize}
\textbf{Output} \(\text{max}\)

Running time: suppose \(G\) has \(n\) vertices and \(m\) edges
\begin{itemize}
  \item \(2^n\) subsets of \(V\)
  \item checking each subset \(S\) takes \(O(m)\) time
  \item total time is \(O(m2^n)\)
\end{itemize}
Algorithm to find the size of the maximum weight independent set.

\[
\text{MaxIndSet}(G = (V, E)):
\]

\[
\begin{align*}
\text{max} &= 0 \\
\text{for each subset } S \subseteq V \text{ do} \\
&\quad \text{check if } S \text{ is an independent set} \\
&\quad \text{if } S \text{ is an independent set and } w(S) > \text{max} \text{ then} \\
&\quad \quad \text{max} = w(S)
\end{align*}
\]

Output \( \text{max} \)

Running time: suppose \( G \) has \( n \) vertices and \( m \) edges

1. \( 2^n \) subsets of \( V \)
2. checking each subset \( S \) takes \( O(m) \) time
3. total time is \( O(m2^n) \)
A Recursive Algorithm

Let $V = \{v_1, v_2, \ldots, v_n\}$.

For a vertex $u$ let $N(u)$ be its neighbors.

Observation

$v_1$: vertex in the graph.

One of the following two cases is true

- **Case 1** $v_1$ is in some maximum independent set.
- **Case 2** $v_1$ is in no maximum independent set.

We can try both cases to “reduce” the size of the problem

$G_1 = G - v_1$ obtained by removing $v_1$ and incident edges from $G$

$G_2 = G - v_1 - N(v_1)$ obtained by removing $N(v_1) \cup v_1$ from $G$

$$MIS(G) = \max\{MIS(G_1), MIS(G_2) + w(v_1)\}$$
A Recursive Algorithm

Let \( V = \{v_1, v_2, \ldots, v_n\} \).

For a vertex \( u \) let \( N(u) \) be its neighbors.

**Observation**

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\[ \text{MIS}(G) = \max\{\text{MIS}(G_1), \text{MIS}(G_2) + w(v_1)\} \]
A Recursive Algorithm

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For a vertex $u$ let $N(u)$ be its neighbors.

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$$MIS(G) = \max\{MIS(G_1), MIS(G_2) + w(v_1)\}$$
A Recursive Algorithm

\textbf{RecursiveMIS} (\(G\)):
\begin{align*}
\text{if } G \text{ is empty then Output } 0 \\
a &= \text{RecursiveMIS} (G - v_1) \\
b &= w(v_1) + \text{RecursiveMIS} (G - v_1 - N(v_n)) \\
\text{Output } \max(a, b)
\end{align*}
Example
Recursive Algorithms
..for Maximum Independent Set

Running time:

\[ T(n) = T(n - 1) + T\left(n - 1 - \text{deg}(v_1)\right) + O(1 + \text{deg}(v_1)) \]

where \( \text{deg}(v_1) \) is the degree of \( v_1 \). \( T(0) = T(1) = 1 \) is base case.

Worst case is when \( \text{deg}(v_1) = 0 \) when the recurrence becomes

\[ T(n) = 2T(n - 1) + O(1) \]

Solution to this is \( T(n) = O(2^n) \).
Backtrack Search via Recursion

1. Recursive algorithm generates a tree of computation where each node is a smaller problem (subproblem)
2. Simple recursive algorithm computes/explores the whole tree blindly in some order.
3. Backtrack search is a way to explore the tree intelligently to prune the search space
   1. Some subproblems may be so simple that we can stop the recursive algorithm and solve it directly by some other method
   2. Memoization to avoid recomputing same problem
   3. Stop the recursion at a subproblem if it is clear that there is no need to explore further.
   4. Leads to a number of heuristics that are widely used in practice although the worst case running time may still be exponential.
Sequences

**Definition**

**Sequence**: an ordered list $a_1, a_2, \ldots, a_n$. **Length** of a sequence is number of elements in the list.

**Definition**

$a_{i_1}, \ldots, a_{i_k}$ is a **subsequence** of $a_1, \ldots, a_n$ if $1 \leq i_1 < i_2 < \ldots < i_k \leq n$.

**Definition**

A sequence is **increasing** if $a_1 < a_2 < \ldots < a_n$. It is **non-decreasing** if $a_1 \leq a_2 \leq \ldots \leq a_n$. Similarly **decreasing** and **non-increasing**.
Example

1. Sequence: 6, 3, 5, 2, 7, 8, 1, 9
2. Subsequence of above sequence: 5, 2, 1
3. Increasing sequence: 3, 5, 9, 17, 54
4. Decreasing sequence: 34, 21, 7, 5, 1
5. Increasing subsequence of the first sequence: 2, 7, 9.
Longest Increasing Subsequence Problem

**Input**  A sequence of numbers $a_1, a_2, \ldots, a_n$

**Goal**  Find an **increasing subsequence** $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ of maximum length

**Example**

1. **Sequence:** 6, 3, 5, 2, 7, 8, 1
2. **Increasing subsequences:** 6, 7, 8 and 3, 5, 7, 8 and 2, 7 etc
3. **Longest increasing subsequence:** 3, 5, 7, 8
Longest Increasing Subsequence Problem

Input  A sequence of numbers $a_1, a_2, \ldots, a_n$

Goal  Find an increasing subsequence $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ of maximum length

Example

1. Sequence: 6, 3, 5, 2, 7, 8, 1
2. Increasing subsequences: 6, 7, 8 and 3, 5, 7, 8 and 2, 7 etc
3. Longest increasing subsequence: 3, 5, 7, 8
Naïve Enumeration

Assume $a_1, a_2, \ldots, a_n$ is contained in an array $A$

```python
algLISNaive(A[1..n]):
    max = 0
    for each subsequence $B$ of $A$ do
        if $B$ is increasing and $|B| > max$ then
            max = |B|
    Output max
```

Running time: $O(n2^n)$.

$2^n$ subsequences of a sequence of length $n$ and $O(n)$ time to check if a given sequence is increasing.
Naïve Enumeration

Assume $a_1, a_2, \ldots, a_n$ is contained in an array $A$

```
algLISNaive(A[1..n]):
    max = 0
    for each subsequence $B$ of $A$ do
        if $B$ is increasing and $|B| > max$ then
            max = $|B|$
    Output $max$
```

Running time: $O(n2^n)$.
$2^n$ subsequences of a sequence of length $n$ and $O(n)$ time to check if a given sequence is increasing.
Naïve Enumeration

Assume \( a_1, a_2, \ldots, a_n \) is contained in an array \( A \)

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\text{algLISNaive}(A[1..n]) : \\
\quad \text{max} = 0 \\
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\quad \quad \text{if } B \text{ is increasing and } |B| > \text{max} \text{ then} \\
\quad \quad \quad \text{max} = |B|
\]

Output \( \text{max} \)

Running time: \( O(n2^n) \).

\( 2^n \) subsequences of a sequence of length \( n \) and \( O(n) \) time to check if a given sequence is increasing.
Recursive Approach: Take 1

**LIS**: Longest increasing subsequence

Can we find a recursive algorithm for **LIS**?

**LIS**($A[1..n]$):

1. **Case 1**: Does not contain $A[n]$ in which case
   \[ \text{LIS}(A[1..n]) = \text{LIS}(A[1..(n-1)]) \]

2. **Case 2**: contains $A[n]$ in which case **LIS**($A[1..n]$) is not so clear.

**Observation**

For second case we want to find a subsequence in $A[1..(n-1)]$ that is restricted to numbers less than $A[n]$. This suggests that a more general problem is **LIS**$_{smaller}$($A[1..n]$, $x$) which gives the longest increasing subsequence in $A$ where each number in the sequence is less than $x$. 

Recursive Approach: Take 1

**LIS:** Longest increasing subsequence

Can we find a recursive algorithm for **LIS**?

**LIS**\( (A[1..n]) \):

1. Case 1: Does not contain \( A[n] \) in which case
   \[ \text{LIS}(A[1..n]) = \text{LIS}(A[1..(n - 1)]) \]

2. Case 2: contains \( A[n] \) in which case \( \text{LIS}(A[1..n]) \) is not so clear.

**Observation**

For second case we want to find a subsequence in \( A[1..(n - 1)] \) that is restricted to numbers less than \( A[n] \). This suggests that a more general problem is **LIS smaller**\( (A[1..n], x) \) which gives the longest increasing subsequence in \( A \) where each number in the sequence is less than \( x \).
Can we find a recursive algorithm for **LIS**?

**LIS**($A[1..n]$):

1. **Case 1**: Does not contain $A[n]$ in which case
   \[ \text{LIS}(A[1..n]) = \text{LIS}(A[1..(n-1)]) \]
2. **Case 2**: contains $A[n]$ in which case **LIS**($A[1..n]$) is not so clear.

**Observation**

For second case we want to find a subsequence in $A[1..(n-1)]$ that is restricted to numbers less than $A[n]$. This suggests that a more general problem is **LIS_smaller**($A[1..n]$, $x$) which gives the longest increasing subsequence in $A$ where each number in the sequence is less than $x$. 
Recursive Approach: Take 1

**LIS**: Longest increasing subsequence

Can we find a recursive algorithm for **LIS**?

\[
\text{LIS}(A[1..n]):
\]

1. **Case 1**: Does not contain \( A[n] \) in which case
   \[
   \text{LIS}(A[1..n]) = \text{LIS}(A[1..(n - 1)])
   \]
2. **Case 2**: contains \( A[n] \) in which case \( \text{LIS}(A[1..n]) \) is not so clear.

**Observation**

For second case we want to find a subsequence in \( A[1..(n - 1)] \) that is restricted to numbers less than \( A[n] \). This suggests that a more general problem is \( \text{LIS\_smaller}(A[1..n], x) \) which gives the longest increasing subsequence in \( A \) where each number in the sequence is less than \( x \).
Recursive Approach

**LIS\_smaller\((A[1..n], x)\)**: length of longest increasing subsequence in \(A[1..n]\) with all numbers in subsequence less than \(x\)

\[
\begin{align*}
\text{LIS\_smaller}(A[1..n], x): \\
& \quad \text{if (} n = 0 \text{) then return 0} \\
& \quad m = \text{LIS\_smaller}(A[1..(n - 1)], x) \\
& \quad \text{if (} A[n] < x \text{) then} \\
& \quad \quad m = \max(m, 1 + \text{LIS\_smaller}(A[1..(n - 1)], A[n])) \\
\text{Output } m
\end{align*}
\]

**LIS\((A[1..n])\)**:

return LIS\_smaller\((A[1..n], \infty)\)
Example

Sequence: $A[1..7] = 6, 3, 5, 2, 7, 8, 1$