For each of the following languages over the alphabet $\Sigma = \{0, 1\}$, either prove the language is regular (by giving an equivalent regular expression, DFA, or NFA) or prove that the language is not regular (using a fooling set argument). Exactly half of these languages are regular.

1. $\{0^n10^n \mid n \geq 0\}$

**Solution:**

Not regular: Any two strings $x = 0^i$ and $y = 0^j$ are distinguished by the suffix $z = 10^i$. Thus, $0^*$ is a fooling set.

2. $\{0^n10^nw \mid n \geq 0 \text{ and } w \in \Sigma^*\}$

**Solution:**

Not regular. Any two strings $x = 0^i$ and $y = 0^j$ where $i < j$ are distinguished by the suffix $z = 10^i$. (It is crucial that $i < j$ here!) Thus, $0^*$ is a fooling set.

3. $X = \{w0^n10^n x \mid w \in \Sigma^* \text{ and } n \geq 0 \text{ and } x \in \Sigma^*\}$

**Solution:**

Regular. We might as well set $n = 0$, since any bigger value can be absorbed by the attached $w$ and $x$. Namely, we have

$$X = \{w0^n10^n x \mid w \in \Sigma^* \text{ and } n \geq 0 \text{ and } x \in \Sigma^*\} = \{w1x \mid w, x \in \Sigma^*\}.$$

This is the set of all strings containing the symbol 1, which is described by the regular expression $(0 + 1)^*1(0 + 1)^*$.

4. Strings in which the number of 0s and the number of 1s differ by at most 2.

**Solution:**

Not regular. Any two strings $x = 0^i$ and $y = 0^j$ where $i < j$ are distinguished by the suffix $z = 1^{j+2}$. (It is crucial that $i < j$ here!) Thus, $0^*$ is a fooling set.

5. Strings such that in every prefix, the number of 0s and the number of 1s differ by at most 2.

**Solution:**

Regular. Keep track of the difference between the number of 0s and the number of 1s seen so far. If this difference is ever less than $-2$ or greater than 2, reject; otherwise, accept. So we get a six-state DFA, where five of the states are accepting.

6. Strings such that in every substring, the number of 0s and the number of 1s differ by at most 2.
Solution:

Regular. Keep track of the *current* difference between the number of 0s and the number of 1s seen so far. Also keep track of the *maximum* and *minimum* value of this difference seen so far. If the max-difference is ever more than min-difference+2, reject. Crudely, there are at most 45 possible values of (curr-dif, max-diff, min-diff), so we get a DFA with at most 46 states.

Alternatively, we can non-deterministically guess the range of differences (−2 ≤ diff ≤ 0 or −1 ≤ diff ≤ 1 or 0 ≤ diff ≤ 2), build a separate DFA for each guess, and combine the three DFAs into a single 10-state NFA.