Prove that each of the following languages is not regular.

1. \( \{ 0^{2n} \mid n \geq 0 \} \)

**Solution:**

Let \( F = L = \{ 0^{2n} \mid n \geq 0 \} \).

Let \( x \) and \( y \) be arbitrary elements of \( F \).

Then \( x = 0^x \) and \( y = 0^y \) for some non-negative integers \( x \) and \( y \).

Let \( z = 0^{2^y} \).

Then \( xz = 0^{2^x}0^{2^y} = 0^{2^x+2^y} \in L \).

And \( yz = 0^{2^x}0^{2^y} = 0^{2^x+2^y} \not \in L \), because \( i \neq j \)

Thus, \( F \) is a fooling set for \( L \).

Because \( F \) is infinite, \( L \) cannot be regular.

2. \( \{ 0^{2n}1^n \mid n \geq 0 \} \)

**Solution:**

Let \( F \) be the language \( 0^* \).

Let \( x \) and \( y \) be arbitrary strings in \( F \).

Then \( x = 0^i \) and \( y = 0^j \) for some non-negative integers \( i \neq j \).

Let \( z = 0^i \).

Then \( xz = 0^{2^i}1^i \in L \).

And \( yz = 0^{i+j}1^i \not \in L \), because \( i+j \neq 2i \).

Thus, \( F \) is a fooling set for \( L \).

Because \( F \) is infinite, \( L \) cannot be regular.

3. \( \{ 0^n1^n \mid m \neq 2n \} \)

**Solution:**

Let \( F \) be the language \( 0^* \).

Let \( x \) and \( y \) be arbitrary strings in \( F \).

Then \( x = 0^i \) and \( y = 0^j \) for some non-negative integers \( i \neq j \).

Let \( z = 0^i \).

Then \( xz = 0^{2i}1^i \not \in L \).

And \( yz = 0^{i+j}1^i \in L \), because \( i+j \neq 2i \).
Thus, $F$ is a fooling set for $L$.
Because $F$ is infinite, $L$ cannot be regular.

**Solution:**

For all non-negative integers $i \neq j$, the strings $0^{2i}$ and $0^{2j}$ are distinguished by the suffix $1^i$, because $0^{2i+1} \notin L$ but $0^{2j+1} \in L$. Thus, the language $(00)^*$ is an infinite fooling set for $L$.

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4. Strings over $\{0, 1\}$ where the number of 0s is exactly twice the number of 1s.

**Solution:**

Let $F$ be the language $0^*$. Let $x$ and $y$ be arbitrary strings in $F$.
Then $x = 0^i$ and $y = 0^j$ for some non-negative integers $i \neq j$.
Let $z = 0^i 1^j$.
Then $xz = 0^{2i} 1^j \in L$.
And $yz = 0^{i+j} 1^j \notin L$, because $i+j \neq 2i$.
Thus, $F$ is a fooling set for $L$.
Because $F$ is infinite, $L$ cannot be regular.

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5. Strings of properly nested parentheses $()$, brackets $[]$, and braces $\{}\{}$. For example, the string $([[]])$ is in this language, but the string $([[]])$ is not, because the left and right delimiters don’t match.

**Solution:**

Let $F$ be the language $^*$. Let $x$ and $y$ be arbitrary strings in $F$.
Then $x = i^i$ and $y = j^j$ for some non-negative integers $i \neq j$.
Let $z = j^j$.
Then $xz = i^i j^j \in L$.
And $yz = i^i j^j \notin L$, because $i \neq j$.
Thus, $F$ is a fooling set for $L$.
Because $F$ is infinite, $L$ cannot be regular.

**Solution:**

For any non-negative integers $i \neq j$, the strings $i^i$ and $j^j$ are distinguished by the suffix $j^j$, because $i^i j^j \in L$ but $i^j j^i \notin L$. Thus, the language $^*$ is an infinite fooling set.
6. Strings of the form \( w_1 \# w_2 \# \cdots \# w_n \) for some \( n \geq 2 \), where each substring \( w_i \) is a string in \( \{0, 1\}^* \), and some pair of substrings \( w_i \) and \( w_j \) are equal.

**Solution:**

Let \( F \) be the language \( 0^* \).

Let \( x \) and \( y \) be arbitrary strings in \( F \).

Then \( x = 0^i \) and \( y = 0^j \) for some non-negative integers \( i \neq j \).

Let \( z = \#0^i \).

Then \( xz = 0^i \#0^i \in L \).

And \( yz = 0^j \#0^i \notin L \), because \( i \neq j \).

Thus, \( F \) is a fooling set for \( L \). Because \( F \) is infinite, \( L \) cannot be regular.

**Solution:**

For any non-negative integers \( i \neq j \), the strings \( 0^i \) and \( 0^j \) are distinguished by the suffix \( \#0^i \), because \( 0^i \#0^i \in L \) but \( 0^j \#0^i \notin L \). Thus, the language \( 0^* \) is an infinite fooling set.

**Extra problems**

7. \( \{0^{n^2} \mid n \geq 0\} \)

**Solution:**

Let \( x \) and \( y \) be distinct arbitrary strings in \( L \).

Without loss of generality, \( x = 0^i^2 \) and \( y = 0^j^2 \) for some \( i > j \geq 0 \).

Let \( z = 0^{2i+1} \).

Then \( xz = 0^{i^2+2i+1} = 0^{(i+1)^2} \in L \).

On the other hand, \( yz = 0^{i^2+2j+1} \notin L \), because \( i^2 < i^2 + 2j + 1 < (i+1)^2 \).

Thus, \( z \) distinguishes \( x \) and \( y \).

We conclude that \( L \) is an infinite fooling set for \( L \), so \( L \) cannot be regular.

**Solution:**

Let \( x \) and \( y \) be distinct arbitrary strings in \( 0^* \).

Without loss of generality, \( x = 0^i \) and \( y = 0^j \) for some \( i > j \geq 0 \).

Let \( z = 0^{i^2+i+1} \).

Then \( xz = 0^{i^2+2i+1} = 0^{(i+1)^2} \in L \).

On the other hand, \( yz = 0^{i^2+i+j+1} \notin L \), because \( i^2 < i^2 + i + j + 1 < (i+1)^2 \).

Thus, \( z \) distinguishes \( x \) and \( y \).

We conclude that \( 0^* \) is an infinite fooling set for \( L \), so \( L \) cannot be regular.

**Solution:**

Let \( x \) and \( y \) be distinct arbitrary strings in \( 0000^* \).

Without loss of generality, \( x = 0^i \) and \( y = 0^j \) for some \( i > j \geq 3 \).

Let \( z = 0^{i^2-i} \).

Then \( xz = 0^i^2 \in L \).

On the other hand, \( yz = 0^{i^2-i+j} \notin L \), because \( (i-1)^2 = i^2 - 2i + 1 < i^2 - i < i^2 - i + j < i^2 \).
(The first inequalities requires \( i \geq 2 \), and the second \( j \geq 1 \).) Thus, \( z \) distinguishes \( x \) and \( y \).

We conclude that 0000* is an infinite fooling set for \( L \), so \( L \) cannot be regular.

8. \( \{ w \in (0 + 1)^* \mid w \) is the binary representation of a perfect square\}

Solution:

We design our fooling set around numbers of the form \((2^k + 1)^2 = 2^{2k} + 2^{k+1} + 1 = 10^{k-2}10^k1 \in L\), for any integer \( k \geq 2 \). The argument is somewhat simpler if we further restrict \( k \) to be even.

Let \( F = 1(00)^*1 \), and let \( x \) and \( y \) be arbitrary strings in \( F \).

Then \( x = 10^{2i-1} \) and \( y = 10^{2j-1} \), for some positive integers \( i \neq j \).

Without loss of generality, assume \( i < j \). (Otherwise, swap \( x \) and \( y \).)

Let \( z = 0^{2i}1 \).

Then \( xz = 10^{2i-2}10^{2i}1 \) is the binary representation of \( 2^{2i} + 2^i + 1 = (2^i + 1)^2 \), and therefore \( xz \in L \).

On the other hand, \( yz = 10^{2j-2}10^{2j}1 \) is the binary representation of \( 2^{2j} + 2^{i+1} + 1 \). Simple algebra gives us the inequalities

\[
(2^{i+j})^2 = 2^{2i+2j} \\
< 2^{2i+2j} + 2^{2i+1} + 1 \\
< 2^{2(i+j)} + 2^{i+j+1} + 1 \\
= (2^{i+j} + 1)^2.
\]

So \( 2^{2i+2j} + 2^{2i+1} + 1 \) lies between two consecutive perfect squares, and thus is not a perfect square, which implies that \( yz \notin L \).

We conclude that \( F \) is a fooling set for \( L \). Because \( F \) is infinite, \( L \) cannot be regular.