Prove that each of the following problems is NP-hard.

1. Given an undirected graph $G$, does $G$ contain a simple path that visits all but 374 vertices?

2. Given an undirected graph $G$, does $G$ have a spanning tree in which every node has degree at most 374?

3. Given an undirected graph $G$, does $G$ have a spanning tree with at most 374 leaves?

4. Recall that a 5-coloring of a graph $G$ is a function that assigns each vertex of $G$ a “color” from the set $\{0, 1, 2, 3, 4\}$, such that for any edge $uv$, vertices $u$ and $v$ are assigned different “colors”. A 5-coloring is careful if the colors assigned to adjacent vertices are not only distinct, but differ by more than $1 \pmod{5}$. Prove that deciding whether a given graph has a careful 5-coloring is NP-hard. (Hint: Reduce from the standard 5COLOR problem.)

![Figure 1: A careful 5-coloring.](image)

5. Prove that the following problem is NP-hard: Given an undirected graph $G$, find any integer $k > 374$ such that $G$ has a proper coloring with $k$ colors but $G$ does not have a proper coloring with $k - 374$ colors.

6. To think about later: A bicoloring of an undirected graph assigns each vertex a set of two colors. There are two types of bicoloring: In a weak bicoloring, the endpoints of each edge must use different sets of colors; however, these two sets may share one color. In a strong bicoloring, the endpoints of each edge must use distinct sets of colors; that is, they must use four colors altogether. Every strong bicoloring is also a weak bicoloring.

   (a) Prove that finding the minimum number of colors in a weak bicoloring of a given graph is NP-hard.
   (b) Prove that finding the minimum number of colors in a strong bicoloring of a given graph is NP-hard.
Recall that a $\kappa$-coloring of a graph $G$ is a function that assigns each vertex of $G$ a "color" from the set \{0, 1, 2, 3, 4\}, such that for any edge $uv$, vertices $u$ and $v$ are assigned different "colors". A $\kappa$-coloring is careful if the colors assigned to adjacent vertices are not only distinct, but differ by more than 1 (mod 5).

Prove that deciding whether a given graph has a careful 5-coloring is NP-hard.

[Hint: Reduce from the standard \texttt{C} problem.]

A careful 5-coloring.

Prove that the following problem is NP-hard: Given an undirected graph $G$, find any integer $k > 374$ such that $G$ has a proper coloring with $k$ colors but $G$ does not have a proper coloring with $k+374$ colors.

A bicoloring of an undirected graph assigns each vertex a set of two colors. There are two types of bicoloring: In a weak bicoloring, the endpoints of each edge must use different sets of colors; however, these two sets may share one color. In a strong bicoloring, the endpoints of each edge must use distinct sets of colors; that is, they must use four colors altogether. Every strong bicoloring is also a weak bicoloring.

(a) Prove that finding the minimum number of colors in a weak bicoloring of a given graph is NP-hard.

(b) Prove that finding the minimum number of colors in a strong bicoloring of a given graph is NP-hard.

Figure 2: Left: A weak bicoloring of a 5-clique with four colors. Right A strong bicoloring of a 5-cycle with five colors.