The following problems ask you to prove some obvious claims about recursively-defined string functions. In each case, we want a self-contained, step-by-step induction proof that builds on formal definitions and prior results, not on intuition. In particular, your proofs must refer to the formal recursive definitions of string length and string concatenation:

\[
|w| := \begin{cases} 
0 & \text{if } w = \varepsilon \\
1 + |x| & \text{if } w = ax \text{ for some symbol } a \text{ and some string } x
\end{cases}
\]

\[
w \cdot z := \begin{cases} 
z & \text{if } w = \varepsilon \\
a \cdot (x \cdot z) & \text{if } w = ax \text{ for some symbol } a \text{ and some string } x
\end{cases}
\]

You may freely use the following results, which are proved in the lecture notes:

**Lemma 1:** \(w \cdot \varepsilon = w\) for all strings \(w\).

**Lemma 2:** \(|w \cdot x| = |w| + |x|\) for all strings \(w\) and \(x\).

**Lemma 3:** \((w \cdot x) \cdot y = w \cdot (x \cdot y)\) for all strings \(w, x,\) and \(y\).

The **reversal** \(w^R\) of a string \(w\) is defined recursively as follows:

\[
w^R := \begin{cases} 
\varepsilon & \text{if } w = \varepsilon \\
x^R \cdot a & \text{if } w = ax \text{ for some symbol } a \text{ and some string } x
\end{cases}
\]

For example, \(\text{STRESSED}^R = \text{DESSERTS}\) and \(\text{WTF374}^R = 473\text{FTW}\).

1. Prove that \(|w| = |w^R|\) for every string \(w\).
2. Prove that \((w \cdot z)^R = z^R \cdot w^R\) for all strings \(w\) and \(z\).
3. Prove that \((w^R)^R = w\) for every string \(w\).

**Hint:** You need \#2 to prove \#3, but you may find it easier to solve \#3 first.

**To think about later:** Let \(#(a, w)\) denote the number of times symbol \(a\) appears in string \(w\). For example, \(#(X, \text{WTF374}) = 0\) and \(#(0, 000010101010010100) = 12\).

4. Give a formal recursive definition of \(#(a, w)\).
5. Prove that \(#(a, w \cdot z) = #(a, w) + #(a, z)\) for all symbols \(a\) and all strings \(w\) and \(z\).
6. Prove that \(#(a, w^R) = #(a, w)\) for all symbols \(a\) and all strings \(w\).