1. (100 PTS.) Draw me a sheep.

For each of the following languages, draw (or describe formally) an NFA that accepts them. Your automata should have a small number of states. Provide a short explanation of your solution, if needed.

(A) All strings over $\{a, b, c\}^*$ in which every nonempty maximal substring of consecutive as is of even length.

(B) $\Sigma^* a \Sigma^* b \Sigma^* c \Sigma^*$.

(C) $(a(a+b)^*a + b(b+c)^*b + c(c+a)^*c)^*$.

(D) $\Big(\big((aa + aab)^*(bab + bb)^* + c\big)\, b\Big)^* + bb$.

(E) All strings in 1^* of length that is divisible by at lease one of the following numbers $2, 3, 5, 7$. For full credit your automata should have less than (say) 20 states.

(F) All strings in a^* of length that is NOT divisible by any of the following numbers $2, 3, 5, 7$.

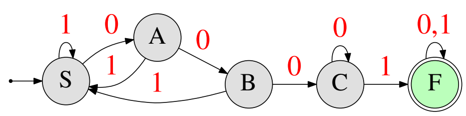
2. (100 PTS.) Blip blop.

For two binary strings $x, y \in \{0, 1\}^*$, of the same length, their ***Hamming distance*** $d_H(x, y)$ is the number of bits in which they differ. For example $d_H(1111, 1111) = 0$, $d_H(0001, 1111) = 3$, and $d_H(1111001, 1111011) = 1$. As a negative example, observe that $d_H(11, 1011)$ is not defined.

Let $L \subseteq \{0, 1\}^*$ be a regular language.

(A) Consider the language $L_{\leq 1} = \{x \in \{0, 1\}^* \mid \exists y \in L \text{ s.t. } d_H(x, y) \leq 1\}$. Describe in words what the language $L_{\leq 1}$ is.

(B) Consider the following DFA M.



What is its language $L = L(M)$?

(C) By modifying the given DFA give above, describe an NFA that that accepts the language $L_{\leq 1}$. Explain your construction.

(D) More generally, demonstrate that if a language $L \subseteq \{0, 1\}^*$ is regular, then $L_{\leq 1}$ is a regular language (for simplicity, you can assume $\varepsilon \notin L$). Specifically, consider a DFA for L, and describe in detail how to modify it to an NFA for $L_{\leq 1}$. (The description of the NFA does not have to be formal here.) Explain why the constructed NFA accept the desired language.

(E) Prove, that for any constant k, the language $L_{\leq k}$ is regular. Your proof has to be formal and provide all necessary details. (I.e., you need to provide an explicit formal description of the resulting NFA for the new language, and prove that the NFA accepts the language $L_{\leq k}$).

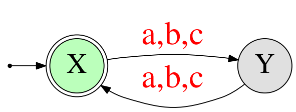
3. (100 PTS.) Codes.

Let Σ be finite alphabet. A ***code*** is a mapping $f : \Sigma \to \{0, 1\}^+$. For example, if $\Sigma = \{a, b, c\}$, a code f might be $f(a) = 00010$, $f(b) = 000$, and $f(c) = 1$. (To simplify things, we assume that $f(a) \neq \varepsilon$, for any character $a \in \Sigma$.)

For a string $w_1 w_2 \cdots w_m \in \Sigma^*$, we define $f(w) = f(w_1) f(w_2) \cdots f(w_m)$. In the above code, we have

$$f(abcba) = 00010 \bullet 000 \bullet 1 \bullet 000 \bullet 00010. = 00010000100000010.$$

(A) (10 PTS.) Let L be the language of the following DFA M. What is L?



(B) (20 PTS.) Working directly on the DFA M from (A) construct an NFA for the language $f(L)$. Here $f(L) = \{f(w) \mid w \in L\}$ is the *code language*. Where f is code from the above example.

(C) (30 PTS.) Let $L \subseteq \Sigma^*$ be am arbitrary regular language. Prove that the encoded language $f(L) = \{f(w) \mid w \in L\}$ is regular.

Specifically, given a DFA $M = (Q, \Sigma, \delta, s, A)$ for L, describe how to build an NFA M' for $f(L)$. Give an upper bound on the number of states of M'.
(I.e., You need to prove the correctness of your construction – that the language of the constructed NFA is indeed the desired language $f(L)$.)
(Rubric: Half the credit is for a correct construction, and the other half is for a correct proof of correctness.)

(D) (40 PTS.) Let $L \subseteq \{0, 1\}^*$ be a regular language. Consider the decoded language $L_f = \{w \in \Sigma^* \mid f(w) \in L\}$. Prove that L_f is a regular language. As above, given a DFA M for L, describe how to construct an NFA for L_f.
(Rubric: Half the credit is for a correct construction, and the other half is for a correct proof of correctness.)