Each homework assignment will include at least one solved problem, similar to the problems assigned in that homework, together with the grading rubric we would apply if this problem appeared on a homework or exam. These model solutions illustrate our recommendations for structure, presentation, and level of detail in your homework solutions. Of course, the actual content of your solutions won’t match the model solutions, because your problems are different!

Solved Problems

4. Recall that the reversal $w^R$ of a string $w$ is defined recursively as follows:

$$w^R := \begin{cases} 
\epsilon & \text{if } w = \epsilon \\
 x^R \cdot a & \text{if } w = a \cdot x 
\end{cases}$$

A palindrome is any string that is equal to its reversal, like AMANAPLANACANALPANAMA, RACECAR, POOP, I, and the empty string.

(a) Give a recursive definition of a palindrome over the alphabet $\Sigma$.

(b) Prove $w = w^R$ for every palindrome $w$ (according to your recursive definition).

(c) Prove that every string $w$ such that $w = w^R$ is a palindrome (according to your recursive definition).

In parts (b) and (c), you may assume without proof that $(x \cdot y)^R = y^R \cdot x^R$ and $(x^R)^R = x$ for all strings $x$ and $y$.

Solution:

(a) A string $w \in \Sigma^*$ is a palindrome if and only if either

- $w = \epsilon$, or
- $w = a$ for some symbol $a \in \Sigma$, or
- $w = axa$ for some symbol $a \in \Sigma$ and some palindrome $x \in \Sigma^*$.

Rubric: 2 points = 1/2 for each base case + 1 for the recursive case. No credit for the rest of the problem unless this is correct.

(b) Let $w$ be an arbitrary palindrome.

Assume that $x = x^R$ for every palindrome $x$ such that $|x| < |w|$.

There are three cases to consider (mirroring the three cases in the definition):

- If $w = \epsilon$, then $w^R = \epsilon$ by definition, so $w = w^R$.
- If $w = a$ for some symbol $a \in \Sigma$, then $w^R = a$ by definition, so $w = w^R$.
Suppose \( w = axa \) for some symbol \( a \in \Sigma \) and some palindrome \( x \in P \). Then

\[
w^R = (a \cdot x \cdot a)^R \\
= (x \cdot a)^R \cdot a \\
= a^R \cdot x^R \cdot a \\
= a \cdot x^R \cdot a \\
= a \cdot x \cdot a \\
= w 
\]

by definition of reversal

You said we could assume this.

by definition of reversal

by the inductive hypothesis

by assumption

In all three cases, we conclude that \( w = w^R \).

Rubric: 4 points: standard induction rubric (scaled)

(c) Let \( w \) be an arbitrary string such that \( w = w^R \).

Assume that every string \( x \) such that \( |x| < |w| \) and \( x = x^R \) is a palindrome.

There are three cases to consider (mirroring the definition of “palindrome”):

- If \( w = \epsilon \), then \( w \) is a palindrome by definition.
- If \( w = a \) for some symbol \( a \in \Sigma \), then \( w \) is a palindrome by definition.
- Otherwise, we have \( w = ax \) for some symbol \( a \) and some non-empty string \( x \).

The definition of reversal implies that \( w^R = (ax)^R = x^Ra \).

Because \( x \) is non-empty, its reversal \( x^R \) is also non-empty.

Thus, \( x^R = by \) for some symbol \( b \) and some string \( y \).

It follows that \( w^R = bya \), and therefore \( w = (w^R)^R = (bya)^R = ay^Rb \).

[At this point, we need to prove that \( a = b \) and that \( y \) is a palindrome.]

Our assumption that \( w = w^R \) implies that \( bya = ay^Rb \).

The recursive definition of string equality immediately implies \( a = b \).

Because \( a = b \), we have \( w = ay^Ra \) and \( w^R = aya \).

The recursive definition of string equality implies \( y^Ra = ya \).

It immediately follows that \( (y^Ra)^R = (ya)^R \).

Known properties of reversal imply \( (y^Ra)^R = a(y^R)^R = ay \) and \( (ya)^R = ay^R \).

It follows that \( ay^R = ay \), and therefore \( y = y^R \).

The inductive hypothesis now implies that \( y \) is a palindrome.

We conclude that \( w \) is a palindrome by definition.

In all three cases, we conclude that \( w \) is a palindrome.

Rubric: 4 points: standard induction rubric (scaled).

- No penalty for jumping from \( aya = ay^Ra \) directly to \( y = y^R \).
– No credit here if the case analysis omits an infinite number of objects. (For example: all odd-length palindromes.)
– −1 if the case analysis omits an finite number of objects. (For example: the empty string.)
– −1 for making the reader infer the case conditions. Spell them out!
– No penalty if cases overlap (for example:
+ 1 for cases that do not invoke the inductive hypothesis (“base cases”)
  – No credit here if one or more “base cases” are missing.
+ 2 for correctly applying the *stated* inductive hypothesis
  – No credit here for applying a *different* inductive hypothesis, even if that different inductive hypothesis would be valid.
+ 2 for other details in cases that invoke the inductive hypothesis (“inductive cases”)
  – No credit here if one or more “inductive cases” are missing.