Undecidability II

Lecture 26

Example of Undecidable Language

$SELFREJECT$ = { <M> | M rejects <M> }

M =Turing Machine (piece of executable code)

<M> = encoding of M as a string (source code for M)

<M> is what you would feed to a universal TM, that would allow it to simulate M.

(e.g. TM that rejects everything. TM that rejects every description of a TM are in that language)

Showing Undecidability

To show L is undecidable, reduce some undecidable language to L

$SELFHALT = \{ \langle M \rangle \mid M$ halts on $\langle M \rangle \}$

Claim: *SELFHALT* is undecidable

More general looking problem:

$HALT = \{ \langle M,w \rangle \mid M$ halts on $w \}$

Claim: *HALT* is acceptable

The halting problem

Claim: *HALT* is undecidable

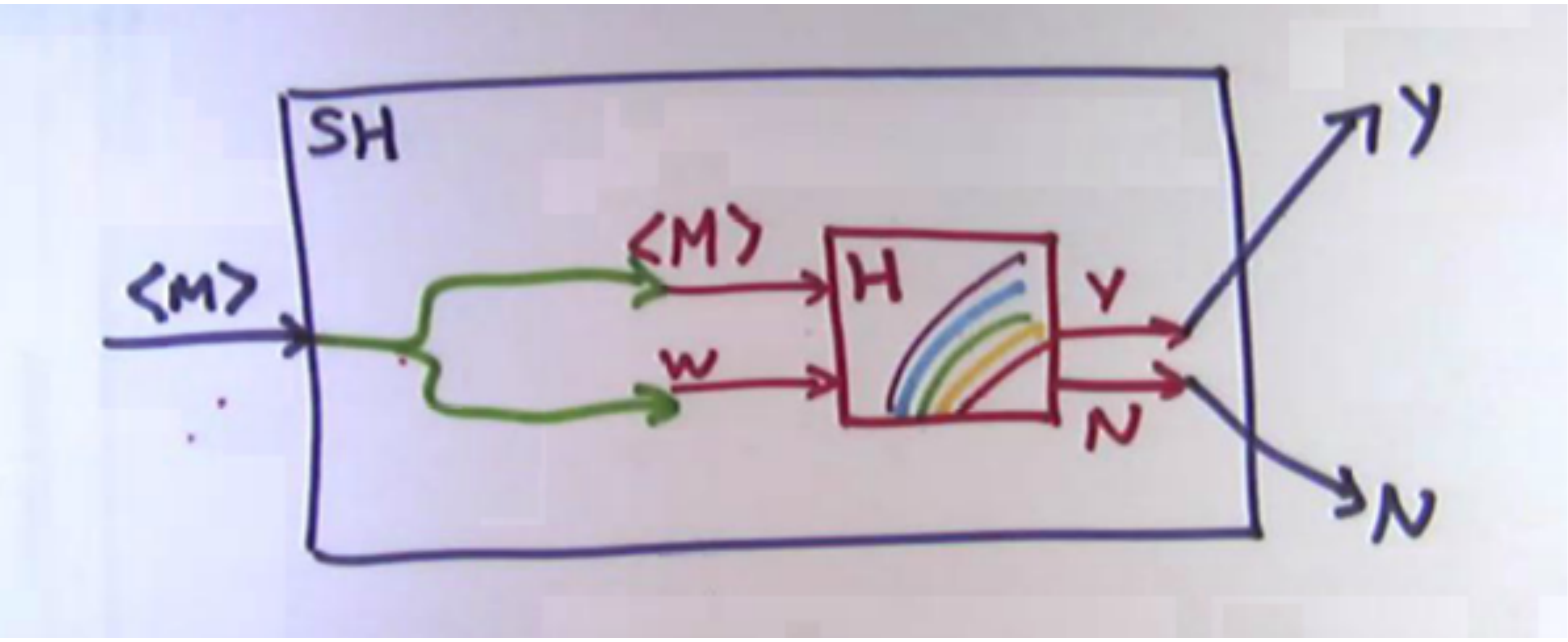
Showing Undecidability

$HALT = \{ \langle M,w \rangle \mid M$ halts on $w\ \}$

Claim: *HALT* is undecidable

Proof:

Suppose (towards contradiction) that there is a TM H that decides HALT. Reduce from SELFHALT



$\text{NEVER}\mathit{ACCEPT} = \{ \langle M \rangle \mid \mathit{ACCEPT(M)}=\emptyset \}$

(is a TM useless or not?)

Claim: NEVERACCEPT is undecidable

How many Turing Machines?

- Fix a TM M and an input w.
- Build a new TM M’ with the following behavior:
- M’ accepts its input iff M accepts w. (toss input out the window)
- Pseudocode :

M’(x)

Run M(w)

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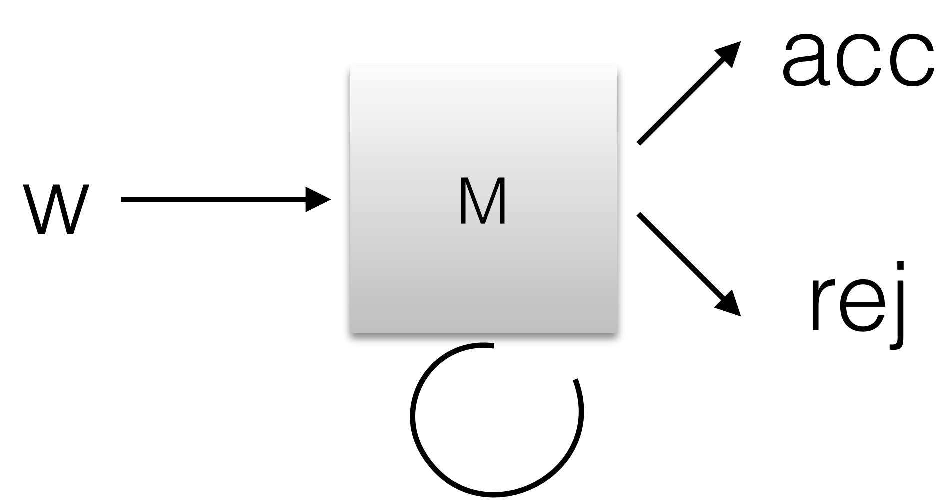

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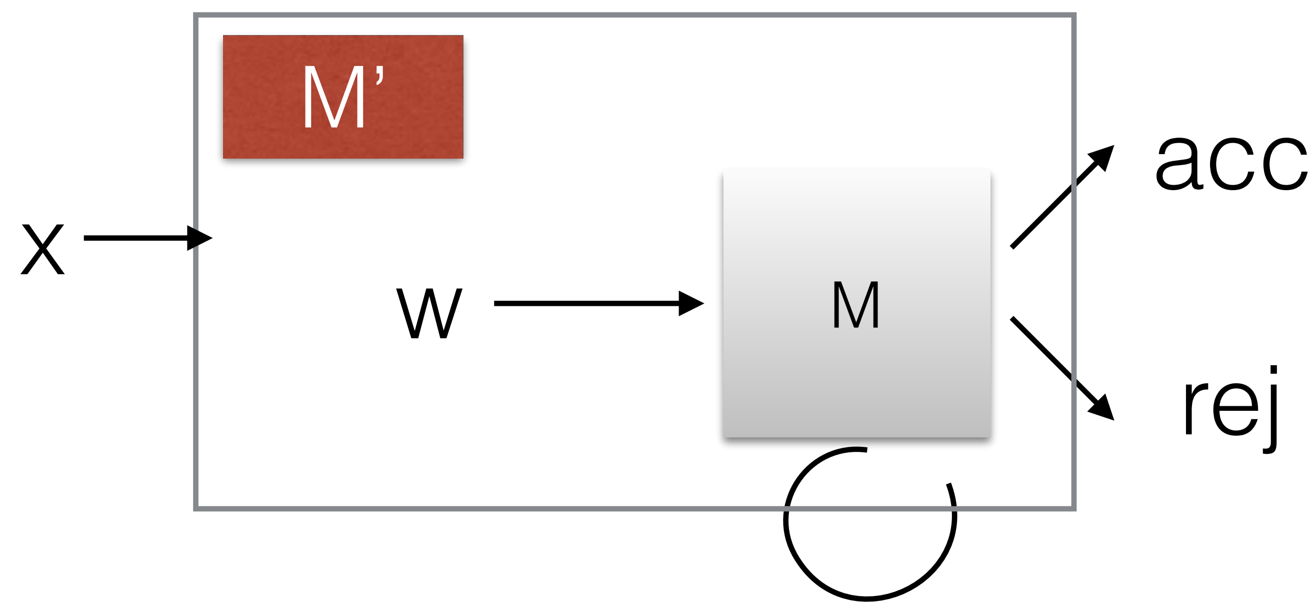


w hardcoded and M hardcoded in M'

- Build M'?

Write a program

Input <M,w>: M - Turing Machine,

w - string

Output <M'>: M' - Turing Machine,

s.t. for any string x, M' accepts x iff M accepts w.

- could produce M' ourselves (write pseudocode).
- So far, when we talk about reduction, WE are doing the reduction
- Now, we need to describe how to do this transformation by writing code that performs the transformation

$$\text{NEVER}\mathit{ACCEPT} = \{ \langle M\rangle \mid \mathit{ACCEPT(M)}=\emptyset \}$$

(M accepts nothing)

Claim: NEVERACCEPT is undecidable

We will assume we know the following:

$$\text{ACCEPT} = \{ \langle M,w\rangle \mid M \textit{ accepts } w \} \text{ is undecidable}$$

Proof:

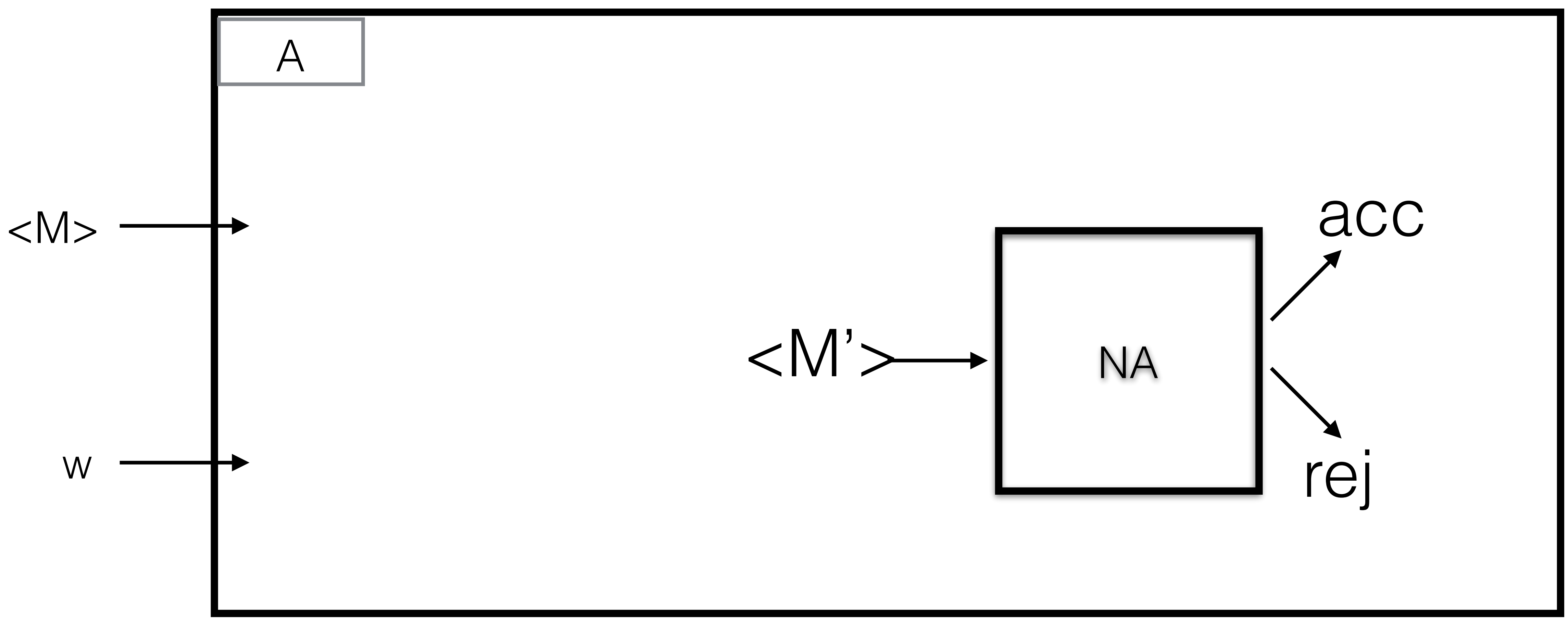
Suppose (towards contradiction) that there is a TM NA that decides NEVERACCEPT.

NEVER*ACCEPT* = { <M> | *ACCEPT(M)=ϕ* }

Claim: NEVERACCEPT is undecidable

Proof:

Suppose (towards contradiction) that there is a TM NA that decides NEVERACCEPT.



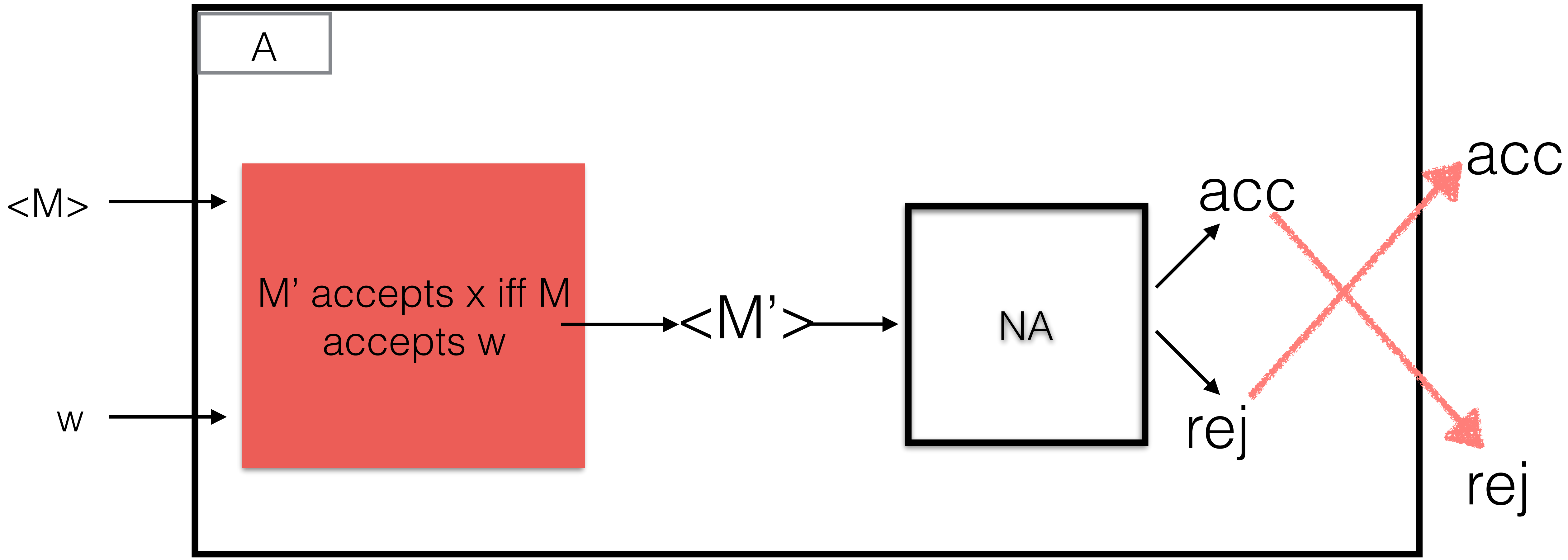
$$NEVERACCEPT = \{ \langle M \rangle \mid ACCEPT(M)=\phi \}$$

Claim: NEVERACCEPT is undecidable

Proof:

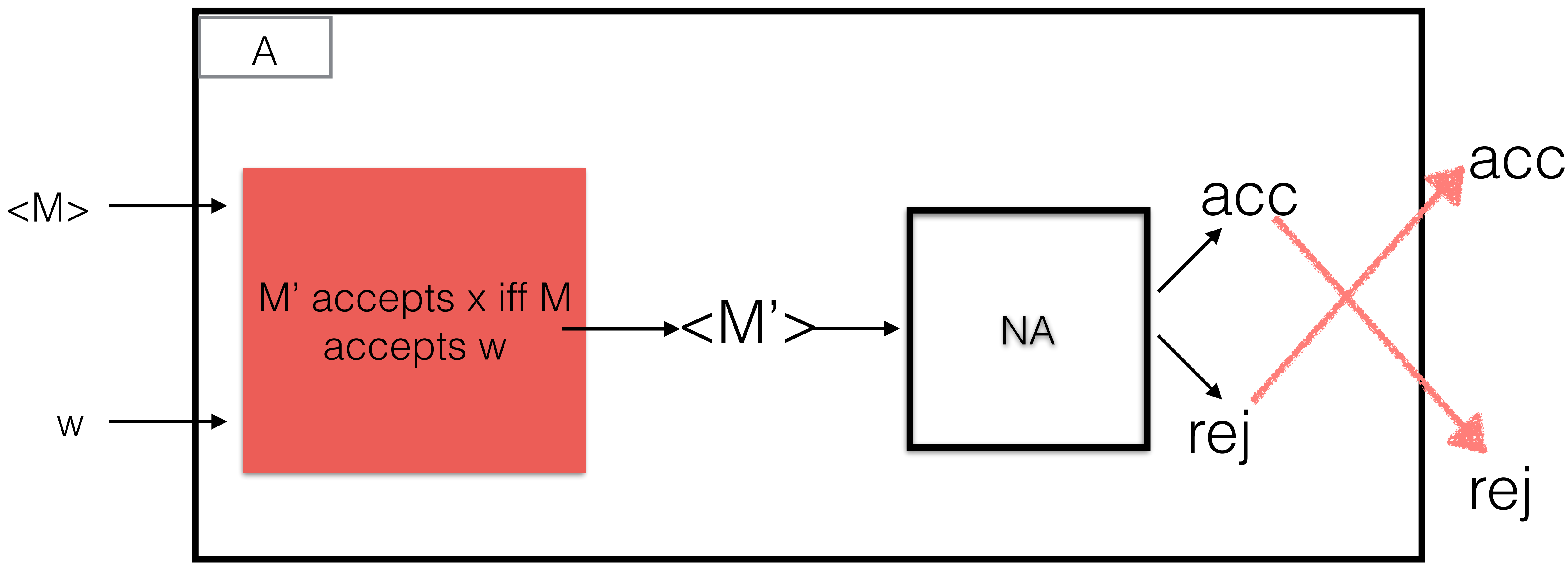
Suppose (towards contradiction) that there is a TM NA that decides NEVERACCEPT.

how many TMs?



when I design a compiler for a piece of code, I can’t worry about the input that this code will be fed many many years from now.

x and w not related!



NEVER$ACCEPT$ = { <M> | $ACCEPT(M)=\phi$ }

Claim: A decides ACCEPT

- **Case 1**: M accepts w.

Implies M’ accepts everything (by def. of M’).

Implies M’ not in NEVERACCEPT (by def of NEVERACCEPT)

Implies NA rejects <M’> (by def of NA)

Implies A accepts <M,w> (by def of A)

$$\text{NEVER}ACCEPT = \{ \langle M \rangle \mid ACCEPT(M)=\phi \}$$

Claim: A decides ACCEPT

- **Case 2**: M doesn’t accept w.

Implies M’ doesn't accept anything (by def. of M’).

Implies M’ in NEVERACCEPT (by def of NEVERACCEPT)

Implies NA accepts <M’> (by def of NA)

Implies A rejects <M,w> (by def of A)

These two cases are exhaustive and imply A decides ACCEPT, contradiction

Rice's Theorem

- We want to answer questions of the form "does the language this machine accepts have some interesting property?"
- L={set of acceptable languages that is not empty and is not the set of all languages}
- e.g. L = set of all languages containing the word "surfing"
- Define ACCEPTIN(L) = {<M>|ACCEPT(M) is in L}
- L =ø : ACCEPTIN(ø) is decidable (always say no, no language is element of ø)
- L= everything: ACCEPTIN(all) is decidable (always say yes: does this TM accept a language?)
- For every other L ACCEPTIN(L) is undecidable

Rice's Theorem

Rice's Theorem. *Let $\mathcal{L}$ be any set of languages that satisfies the following conditions:*

- *There is a Turing machine Y such that* $\textsc{Accept}(Y) \in \mathcal{L}$.
- *There is a Turing machine N such that* $\textsc{Accept}(N) \notin \mathcal{L}$.

The language $\textsc{AcceptIn}(\mathcal{L}) := \left\{ \langle M \rangle \;\middle|\; \textsc{Accept}(M) \in \mathcal{L} \right\}$ *is undecidable.*

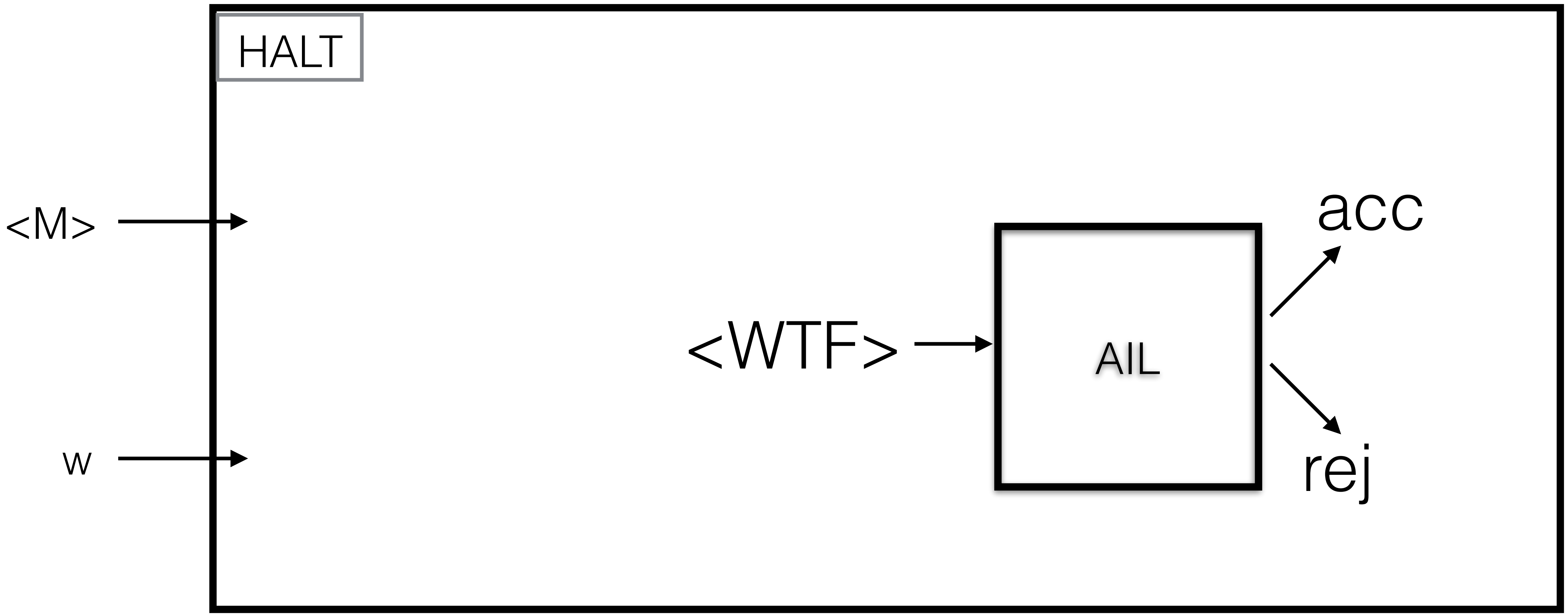
To Show ACCEPTIN(L) is undecidable

Reduce from $HALT$ = { <M,w> | M halts on w }

Rice's Theorem

- ACCEPTIN(L) = {<M>|ACCEPT(M) is in L}

HALT = { <M,w> | *M* halts on w }

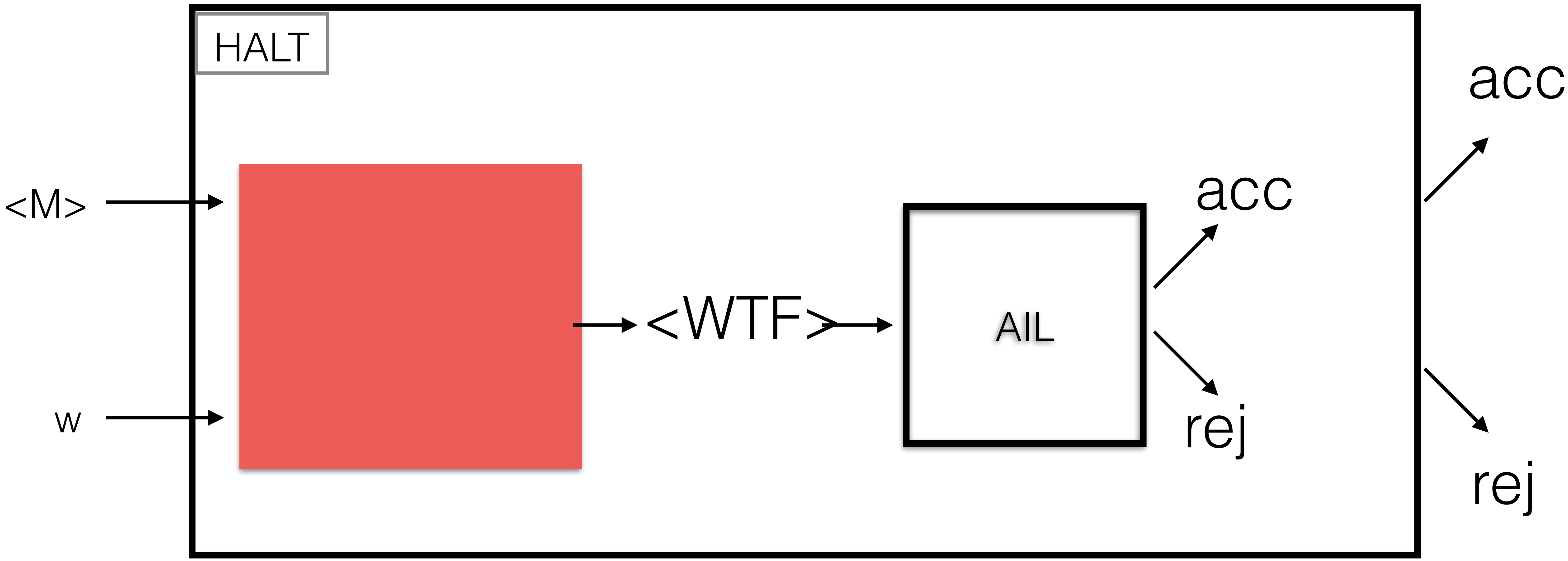


M halts on W iff ACCEPT(WTF) is in L

Rice's Theorem

- ACCEPTIN(L) = {<M>|ACCEPT(M) is in L}

  $HALT$ = { <M,w> | M halts on w }

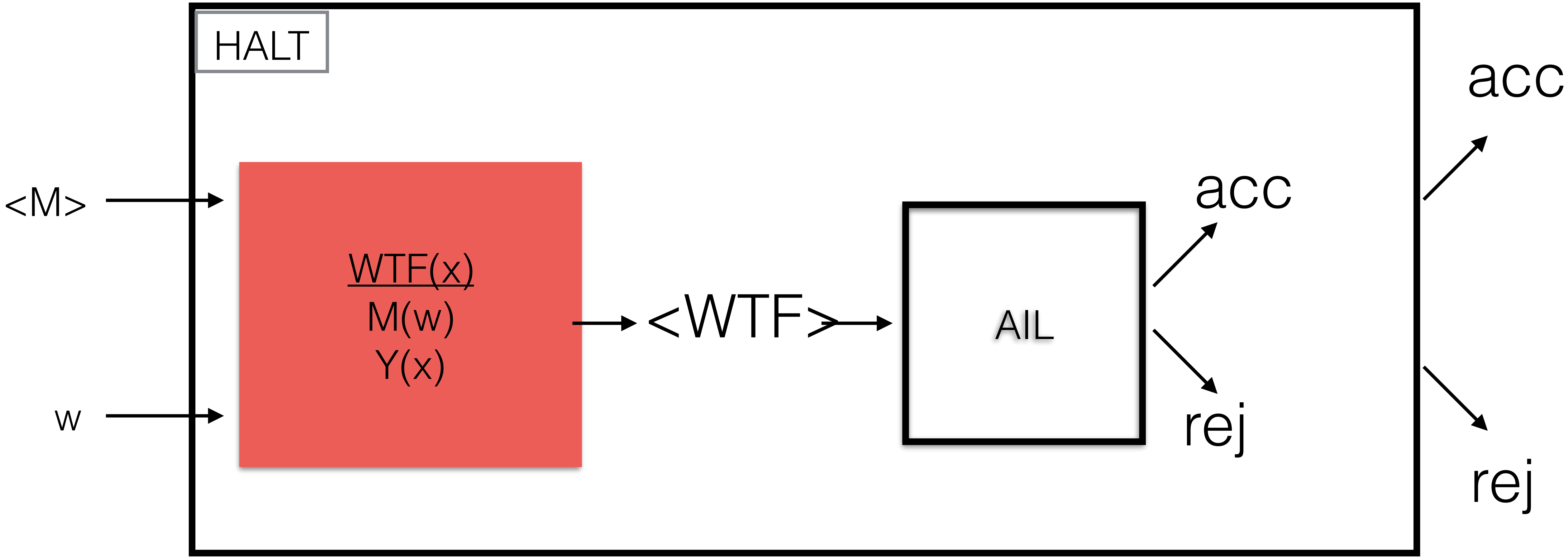


M halts on w iff ACCEPT(WTF) is in L

Rice’s Theorem

- ACCEPTIN(L) = {<M>|ACCEPT(M) is in L}

  $HALT = \{ <M,w> \mid M$ halts on w }

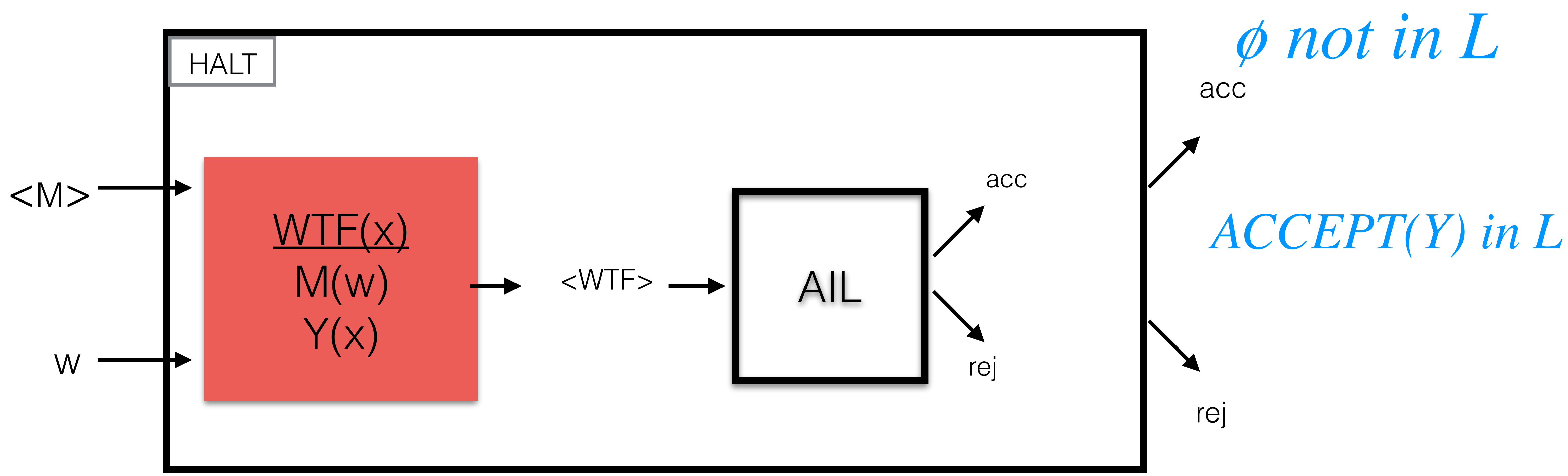


M halts on w iff ACCEPT(WTF) is in L

Assume ϕ not in L. Let Y be a TM so that ACCEPT(Y) in L

Rice's Theorem

- ACCEPTIN(L) = {<M>|ACCEPT(M) is in L}



if *M halts on w then WTF(x) is Y(x) and*

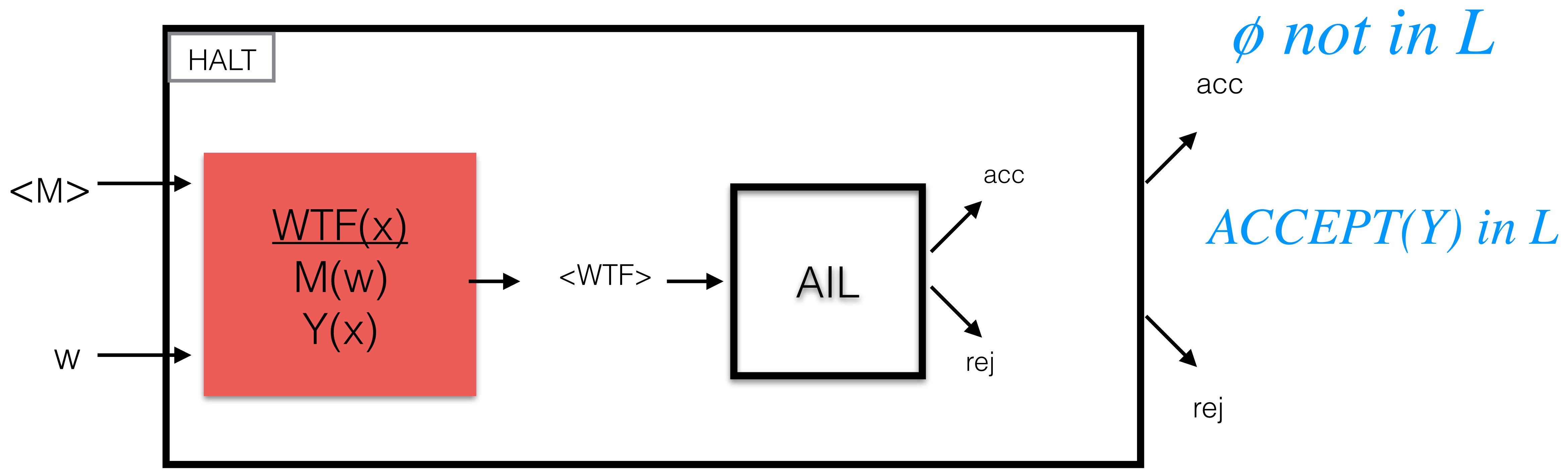
ACCEPT(WTF)=ACCEPT(Y) in L, AIL accepts

if *M doesn't halt on w then WTF(x) never halts*

so ACCEPT(WTF)=ϕ, not in L, AIL rejects

Rice's Theorem

- ACCEPTIN(L) = {<M>|ACCEPT(M) is in L}



H accepts <M,w> iff H halts on w!

contradiction

Rice's Theorem

- example: {<M>| M accepts the empty string}
- M1 accepts nothing : empty string is not in ø
- M2 accepts everything: empty string is in S*
- example: {<M>| M accepts regular language}
- M1 accepts O*
- M2 accepts $\{0^n 1^n : n \geq 0\}$