NP hardness reductions II

Lecture 23
MIN Vertex Cover

- **Input:** a graph $G(V,E)$
- **Output:** Smallest set of vertices that touch every edge

- If $I$ is Independent set in $G$, $V \setminus I$ is vertex cover!
- Largest IS in $G$ is the complement of smallest VC in $G$
what is G’? same graph as G
Output is different
How to prove NP hardness

To prove X is NP-hard:

• **Step 1**: Pick a known NP-hard problem Y

• **Step 2**: Assume for the sake of argument, a polynomial time algorithm for X.

• **Step 3**: Derive a polynomial time algorithm for Y, using algorithm for X as subroutine.

• **Step 4**: Contradiction

Reduce FROM the problem I know about
TO the problem I am curious about

Reduce Y to X
NP hardness of X

• To show X is NP hard (example):

• Poly time reduction from CircuitSAT.

• If there is a poly time algorithm to solve X, then there is poly time algorithm to solve CircuitSAT
NP hardness

- Library of NP-hard problems

- CircuitSAT

- SAT

- 3SAT

- MAX IS

- MAX Clique

- Min Vertex Cover
SAT  
Does a given boolean formula, in CNF, have a satisfying assignment?

3-SAT  
Does a given boolean formula, in CNF with exactly three literals per clause, have a satisfying assignment?

Min Vertex Cover  
In a given undirected graph, what is the (size of the) smallest subset of the vertices covering all of the edges?

Max Independent Set  
In a given undirected graph, what is the (size of the) largest subset of the vertices having no edges in common?

Max Clique  
What is the (size of the) largest complete subgraph of a given undirected graph?

Min Set Cover  
Given a set $S$ and a collection of subsets of $S$, what is smallest set of these subsets whose union is $S$?

Min Hitting Set  
Given a set $S$ and a collection of subsets of $S$, what is smallest subset of $S$ containing at least one element from every subset?

Hamilton Path  
Does a given graph have a Hamilton Path?

Hamilton Cycle  
Does a given graph have a Hamilton Cycle?

Traveling Salesperson  
What is the minimum cost Hamilton Cycle in a weighted, complete, graph?

Longest Path  
What is the longest path between two given nodes in a weighted, undirected, graph?

Subset Sum  
Does a given set of positive integers have a subset with sum $k$?

Partition  
Can a given set of positive integers be partitioned into two subsets each with the same sum?

3-Partition  
Can a given set of $3n$ positive integers be partitioned into $n$ 3-element subsets each with the same sum?

Minesweeper  
In a given Minesweeper configuration, is it safe to click on a particular square?

Sudoku  
Does a given Sodoku puzzle have a solution?
NP hardness

- Library of NP-hard problems
  - CircuitSAT
  - SAT
  - 3SAT
  - MAX IS
  - MAX Clique
  - Min Vertex Cover
  - 3 Coloring
3 Coloring

• Input: a graph $G(V,E)$

• Output: True iff $G$ has a proper 3 coloring

what problem to start with?
3SAT

3CNF Boolean formula

transform in $O(n)$ time

$G$

graph

3Color

TRUE
$G$ is 3-colorable

FALSE
$G$ is not 3-colorable

TRUE
$\Phi$ is satisfiable

FALSE
$\Phi$ is not satisfiable
3COL

- Given an arbitrary 3CNF formula $F$
- Build a graph $G$ as follows
  
  Best described in pieces

  1) piece that corresponds to variables
  2) piece that corresponds to clauses
  3) piece that enforces logical consistency “gadgets”
3COL

- Given an arbitrary 3CNF formula $F$
- Build a graph $G$ as follows
  Best described in pieces
  1) Truth Gadget
3COL

- Given an arbitrary 3CNF formula F
- Build a graph G as follows

Best described in pieces

2) Variable Gadget

one vertex in the graph for every variable and one for its negation. One vertex labeled X
3COL

- Given an arbitrary 3CNF formula F
- Build a graph G as follows

Best described in pieces

3) Clause Gadget
3COL

\[(a \lor b \lor c) \land (b \lor \bar{c} \lor \bar{d}) \land (\bar{a} \lor c \lor d) \land (a \lor \bar{b} \lor \bar{d})\]

in any proper coloring at least one of the three literals must be colored T

easier to prove with 2 SAT example

literal vertices, connected to X
There are 8 possible colorings for the 3 literals on the left.
- For 7 of them one gets colored T and I can properly color the gadget.
- For the 8th, all of them are colored False and I can’t properly color the gadget.
A Boolean network derived from the satisfiable CNF formula:

\[(a \lor b \lor c) \land (b \lor c \lor \bar{d}) \land (a \lor c \lor d) \land (a \lor \bar{b} \lor \bar{d})\]
Proof

Suppose F is satisfiable
So G is 3-Colorable

Suppose G is 3-Colorable
So F is satisfiable
### Proof

<table>
<thead>
<tr>
<th>Suppose F is satisfiable</th>
<th>Suppose G is 3-Colorable</th>
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<tbody>
<tr>
<td>Fix any satisfying assignment</td>
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So G is 3-Colorable

So F is satisfiable
Proof

Suppose $F$ is satisfiable

• Fix any satisfying assignment
• Color True literals same color as $T$
• Color False literals same color as $F$
• By case analysis:
  extend the coloring to the clause gadget

So $G$ is 3-Colorable

Suppose $G$ is 3-Colorable

• Fix a proper 3 Coloring
• Each literal vertex is colored $T$ or $F$
• This gives me an assignment of boolean values to variables
• By case analysis: At least one literal in each clause gadget is colored $T$

So $F$ is satisfiable
4 Coloring?

• Input: a graph $G(V,E)$

• Output: True iff $G$ has a proper 4 coloring
Hamiltonian Cycle

- Input: a directed graph $G(V,E)$
- Output: Is there a cycle in $G$ that visits each vertex exactly once?

- Really asking if there is a way to order the vertices so that every adjacent pair is connected by an edge.

- Reduction from HC if a problem asks for ordering of vertices.

- Anti-topological sort
NP hardness

- Library of NP-hard problems
  - CircuitSAT
  - SAT
  - 3SAT
  - MAX IS
  - MAX Clique
  - Min Vertex Cover
  - 3 Coloring
Hamiltonian Cycle

**Diagram:**

- **VertexCover**
  - Input: $G$ (graph), $k$ (integer)
  - Output: $H$ (graph) transformed in $O(V+E)$ time

- **DirectedHamCycle**
  - Input: $H$ (graph)
  - Output:
    - True: $H$ has a Hamiltonian cycle
    - False: $H$ has no Hamiltonian cycle

- **Rules:**
  - $G$ has a vertex cover of size $k$ if $H$ has a Hamiltonian cycle.
  - $G$ has no vertex cover of size $k$ if $H$ has no Hamiltonian cycle.
Hamiltonian Cycle

• Given an arbitrary graph $G$ and parameter $k$

• Build a graph $H$ as follows

  Best described in gadgets
Hamiltonian Cycle

1) edge gadget

- both $u,v$ in VC
- only $u$ in VC
- only $v$ in VC
Hamiltonian Cycle

2) vertex gadget
Hamiltonian Cycle

2) vertex gadget

connected with edge gadget too
Hamiltonian Cycle

3) cover gadget

\[ 1 \quad 2 \quad 3 \quad k \]