Let \( L \) be an arbitrary regular language.

1. Prove that the language \( \text{insert}_1(L) := \{ x1y \mid xy \in L \} \) is regular.

Intuitively, \( \text{insert}_1(L) \) is the set of all strings that can be obtained from strings in \( L \) by inserting exactly one \( 1 \). For example, if \( L = \{ \epsilon, 00K! \} \), then \( \text{insert}_1(L) = \{ 1, 100K!, 010K!, 001K!, 00K1!, 00K!1 \} \).

2. Prove that the language \( \text{delete}_1(L) := \{ xy \mid x1y \in L \} \) is regular.

Intuitively, \( \text{delete}_1(L) \) is the set of all strings that can be obtained from strings in \( L \) by deleting exactly one \( 1 \). For example, if \( L = \{ 101101, 00, \epsilon \} \), then \( \text{delete}_1(L) = \{ 011101, 10101, 10110 \} \).

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**Work on these later:** (In fact, these might be easier than problems 1 and 2.)

3. Consider the following recursively defined function on strings:

\[
\text{stutter}(w) := \begin{cases} 
\epsilon & \text{if } w = \epsilon \\
ax \cdot \text{stutter}(x) & \text{if } w = ax \text{ for some symbol } a \text{ and some string } x
\end{cases}
\]

Intuitively, \( \text{stutter}(w) \) doubles every symbol in \( w \). For example:

- \( \text{stutter}(\text{PRESTO}) = \text{PPRREESSSTTOO} \)
- \( \text{stutter}(\text{HOCUS} \bullet \text{POCUS}) = \text{HHOOCUSS} \bullet \text{POOCCUSS} \)

Let \( L \) be an arbitrary regular language.

(a) Prove that the language \( \text{stutter}^{-1}(L) := \{ w \mid \text{stutter}(w) \in L \} \) is regular.

(b) Prove that the language \( \text{stutter}(L) := \{ \text{stutter}(w) \mid w \in L \} \) is regular.

4. Consider the following recursively defined function on strings:

\[
\text{evens}(w) := \begin{cases} 
\epsilon & \text{if } w = \epsilon \\
\epsilon & \text{if } w = a \text{ for some symbol } a \\
b \cdot \text{evens}(x) & \text{if } w = abx \text{ for some symbols } a \text{ and } b \text{ and some string } x
\end{cases}
\]

Intuitively, \( \text{evens}(w) \) skips over every other symbol in \( w \). For example:

- \( \text{evens}(\text{EXPELLIARMUS}) = \text{XELAMS} \)
- \( \text{evens}(\text{AVADA} \bullet \text{KEDAVRA}) = \text{VD} \bullet \text{EAR} \).

Once again, let \( L \) be an arbitrary regular language.

(a) Prove that the language \( \text{evens}^{-1}(L) := \{ w \mid \text{evens}(w) \in L \} \) is regular.

(b) Prove that the language \( \text{evens}(L) := \{ \text{evens}(w) \mid w \in L \} \) is regular.