

Proving that a language L is undecidable by reduction requires several steps. (These are the essentially the same steps you already use to prove that a problem is NP-hard.)

- Choose a language L' that you already know is undecidable (because we told you so in class). The simplest choice is usually the standard halting language

$$\text{HALT} := \{ \langle M, w \rangle \mid M \text{ halts on } w \}$$

- Describe an algorithm that decides L' , using an algorithm that decides L as a black box. Typically your reduction will have the following form:

Given an arbitrary string x , construct a special string y ,
such that $y \in L$ if and only if $x \in L'$.

In particular, if $L = \text{HALT}$, your reduction will have the following form:

Given the encoding $\langle M, w \rangle$ of a Turing machine M and a string w ,
construct a special string y , such that
 $y \in L$ if and only if M halts on input w .

- Prove that your algorithm is correct. This proof almost always requires two separate steps:
 - Prove that if $x \in L'$ then $y \in L$.
 - Prove that if $x \notin L'$ then $y \notin L$.

Very important: Name every object in your proof, and *always* refer to objects by their names. Never refer to “the Turing machine” or “the algorithm” or “the input string” or (god forbid) “it” or “this”. Even in casual conversation, even if you’re “just” explaining your intuition, even when you’re just *thinking* about the reduction.

Prove that the following languages are undecidable.

1. $\text{ACCEPTILLINI} := \{ \langle M \rangle \mid M \text{ accepts the string } \mathbf{ILLINI} \}$
2. $\text{ACCEPTTHREE} := \{ \langle M \rangle \mid M \text{ accepts exactly three strings} \}$
3. $\text{ACCEPTPALINDROME} := \{ \langle M \rangle \mid M \text{ accepts at least one palindrome} \}$

Solution (for problem 1): For the sake of argument, suppose there is an algorithm `DECIDEACCEPTILLINI` that correctly decides the language `ACCEPTILLINI`. Then we can solve the halting problem as follows:

<pre> DECIDEHALT($\langle M, w \rangle$): Encode the following Turing machine M': <table border="1"> <tr> <td> <pre> $M'(x)$: run M on input w return TRUE </pre> </td> </tr> </table> if <code>DECIDEACCEPTILLINI($\langle M' \rangle$)</code> return TRUE else return FALSE </pre>	<pre> $M'(x)$: run M on input w return TRUE </pre>
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We prove this reduction correct as follows:

\implies Suppose M halts on input w .

Then M' accepts *every* input string x .

In particular, M' accepts the string **ILLINI**.

So `DECIDEACCEPTILLINI` accepts the encoding $\langle M' \rangle$.

So `DECIDEHALT` correctly accepts the encoding $\langle M, w \rangle$.

\impliedby Suppose M does not halt on input w .

Then M' diverges on *every* input string x .

In particular, M' does not accept the string **ILLINI**.

So `DECIDEACCEPTILLINI` rejects the encoding $\langle M' \rangle$.

So `DECIDEHALT` correctly rejects the encoding $\langle M, w \rangle$.

In both cases, `DECIDEHALT` is correct. But that's impossible, because `HALT` is undecidable. We conclude that the algorithm `DECIDEACCEPTILLINI` does not exist. ■

As usual for undecidability proofs, this proof invokes *four* distinct Turing machines:

- The hypothetical algorithm `DECIDEACCEPTILLINI`.
- The new algorithm `DECIDEHALT` that we construct in the solution.
- The arbitrary machine M whose encoding is part of the input to `DECIDEHALT`.
- The special machine M' whose encoding `DECIDEHALT` constructs (from the encoding of M and w) and then passes to `DECIDEACCEPTILLINI`.