Kartsuba’s Algorithm and Linear Time Selection

Lecture 09
September 22, 2015
Part I

Fast Multiplication
Multiplying Numbers

Problem  Given two $n$-digit numbers $x$ and $y$, compute their product.

Grade School Multiplication

Compute “partial product” by multiplying each digit of $y$ with $x$ and adding the partial products.

\[
\begin{array}{c}
3141 \\
\times 2718 \\
\hline \\
25128 \\
3141 \\
21987 \\
6282 \\
\hline \\
8537238
\end{array}
\]
Each partial product: $\Theta(n)$
Number of partial products: $\Theta(n)$
Addition of partial products: $\Theta(n^2)$
Total time: $\Theta(n^2)$
A Trick of Gauss

Carl Friedrich Gauss: 1777–1855 “Prince of Mathematicians”

Observation: Multiply two complex numbers: \((a + bi)\) and \((c + di)\)

\[
(a + bi)(c + di) = ac - bd + (ad + bc)i
\]
A Trick of Gauss

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Observation: Multiply two complex numbers: \((a + bi)\) and \((c + di)\)

\[(a + bi)(c + di) = ac - bd + (ad + bc)i\]

How many multiplications do we need?
A Trick of Gauss

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Observation: Multiply two complex numbers: \((a + bi)\) and \((c + di)\)

\[(a + bi)(c + di) = ac - bd + (ad + bc)i\]

How many multiplications do we need?

Only 3! If we do extra additions and subtractions. Compute \(ac, bd, (a + b)(c + d)\). Then

\[(ad + bc) = (a + b)(c + d) - ac - bd\]
Divide and Conquer

Assume $n$ is a power of 2 for simplicity and numbers are in decimal.

Split each number into two numbers with equal number of digits

1. $x = x_{n-1}x_{n-2} \ldots x_0$ and $y = y_{n-1}y_{n-2} \ldots y_0$
2. $x = x_{n-1} \ldots x_{n/2}0 \ldots 0 + x_{n/2-1} \ldots x_0$
3. $x_L = 10^{n/2}x_L$ where $x_L = x_{n-1} \ldots x_{n/2}$ and $x_R = x_{n/2-1} \ldots x_0$
4. Similarly $y = 10^{n/2}y_L + y_R$ where $y_L = y_{n-1} \ldots y_{n/2}$ and $y_R = y_{n/2-1} \ldots y_0$
Example

\[1234 \times 5678 = (100 \times 12 + 34) \times (100 \times 56 + 78)\]
\[= 10000 \times 12 \times 56\]
\[+ 100 \times (12 \times 78 + 34 \times 56)\]
\[+ 34 \times 78\]
Divide and Conquer

Assume $n$ is a power of 2 for simplicity and numbers are in decimal.

1. $x = x_{n-1}x_{n-2}\ldots x_0$ and $y = y_{n-1}y_{n-2}\ldots y_0$

2. $x = 10^{n/2}x_L + x_R$ where $x_L = x_{n-1}\ldots x_{n/2}$ and $x_R = x_{n/2-1}\ldots x_0$

3. $y = 10^{n/2}y_L + y_R$ where $y_L = y_{n-1}\ldots y_{n/2}$ and $y_R = y_{n/2-1}\ldots y_0$

Therefore

$$xy = (10^{n/2}x_L + x_R)(10^{n/2}y_L + y_R) = 10^n x_L y_L + 10^{n/2}(x_L y_R + x_R y_L) + x_R y_R$$

$$T(n) = 4T\left(\frac{n}{2}\right) + 6n \quad T(4) = 1$$
Time Analysis

\[ xy = (10^{n/2} x_L + x_R)(10^{n/2} y_L + y_R) \]
\[ = 10^n x_L y_L + 10^{n/2} (x_L y_R + x_R y_L) + x_R y_R \]

4 recursive multiplications of number of size \( n/2 \) each plus 4 additions and left shifts (adding enough 0’s to the right)
\[ xy = (10^{n/2}x_L + x_R)(10^{n/2}y_L + y_R) \]
\[ = 10^n x_L y_L + 10^{n/2}(x_L y_R + x_R y_L) + x_R y_R \]

4 recursive multiplications of number of size \( n/2 \) each plus 4 additions and left shifts (adding enough 0’s to the right)

\[ T(n) = 4T(n/2) + O(n) \quad T(1) = O(1) \]
Time Analysis

\[ xy = (10^{n/2}x_L + x_R)(10^{n/2}y_L + y_R) \]
\[ = 10^n x_L y_L + 10^{n/2}(x_L y_R + x_R y_L) + x_R y_R \]

4 recursive multiplications of number of size \( n/2 \) each plus 4 additions and left shifts (adding enough 0’s to the right)

\[ T(n) = 4T(n/2) + O(n) \quad T(1) = O(1) \]

\[ T(n) = \Theta(n^2) \]. No better than grade school multiplication!
\[ xy = (10^{n/2}x_L + x_R)(10^{n/2}y_L + y_R) \]
\[ = 10^n x_L y_L + 10^{n/2}(x_L y_R + x_R y_L) + x_R y_R \]

4 recursive multiplications of number of size \( n/2 \) each plus 4 additions and left shifts (adding enough 0’s to the right)

\[ T(n) = 4T(n/2) + O(n) \quad T(1) = O(1) \]

\[ T(n) = \Theta(n^2) \] No better than grade school multiplication!

Can we invoke Gauss’s trick here?
Improving the Running Time

\[ xy = (10^{n/2}x_L + x_R)(10^{n/2}y_L + y_R) \]
\[ = 10^n x_L y_L + 10^{n/2}(x_L y_R + x_R y_L) + x_R y_R \]

Gauss trick: \[ x_L y_R + x_R y_L = (x_L + x_R)(y_L + y_R) - x_L y_L - x_R y_R \]

\[ T(n) = 3T\left(\frac{n}{2}\right) + n \]
Improving the Running Time

\[ xy = (10^{n/2}x_L + x_R)(10^{n/2}y_L + y_R) \]
\[ = 10^n x_L y_L + 10^{n/2}(x_L y_R + x_R y_L) + x_R y_R \]

Gauss trick: 
\[ x_L y_R + x_R y_L = (x_L + x_R)(y_L + y_R) - x_L y_L - x_R y_R \]

Recursively compute only \( x_L y_L, x_R y_R, (x_L + x_R)(y_L + y_R) \).
Improving the Running Time

\[ xy = (10^{n/2}x_L + x_R)(10^{n/2}y_L + y_R) \]
\[ = 10^n x_L y_L + 10^{n/2}(x_L y_R + x_R y_L) + x_R y_R \]

Gauss trick: \( x_L y_R + x_R y_L = (x_L + x_R)(y_L + y_R) - x_L y_L - x_R y_R \)

Recursively compute only \( x_L y_L, x_R y_R, (x_L + x_R)(y_L + y_R). \)

**Time Analysis**

Running time is given by

\[ T(n) = 3T(n/2) + O(n) \quad T(1) = O(1) \]

which means
Improving the Running Time

\[ xy = (10^{n/2}x_L + x_R)(10^{n/2}y_L + y_R) \]
\[ = 10^n x_L y_L + 10^{n/2} (x_L y_R + x_R y_L) + x_R y_R \]

Gauss trick: \( x_L y_R + x_R y_L = (x_L + x_R)(y_L + y_R) - x_L y_L - x_R y_R \)

Recursively compute only \( x_L y_L, x_R y_R, (x_L + x_R)(y_L + y_R) \).

Time Analysis

Running time is given by

\[ T(n) = 3T(n/2) + O(n) \quad T(1) = O(1) \]

which means \( T(n) = O(n^{\log_2 3}) = O(n^{1.585}) \)
State of the Art

Schönhage-Strassen 1971: \( O(n \log n \log \log n) \) time using Fast-Fourier-Transform (FFT)

Martin Fürer 2007: \( O(n \log n 2^{O(\log^* n)}) \) time

Conjecture

There is an \( O(n \log n) \) time algorithm.
Analyzing the Recurrences

1. Basic divide and conquer: \( T(n) = 4T(n/2) + O(n), \) \( T(1) = 1. \) **Claim:** \( T(n) = \Theta(n^2). \)

2. Saving a multiplication: \( T(n) = 3T(n/2) + O(n), T(1) = 1. \) **Claim:** \( T(n) = \Theta(n^{1+\log 1.5}) \)
Analyzing the Recurrences

1. Basic divide and conquer: \( T(n) = 4T(n/2) + O(n) \), \( T(1) = 1 \). **Claim:** \( T(n) = \Theta(n^2) \).

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Use recursion tree method:

1. In both cases, depth of recursion \( L = \log n \).

2. Work at depth \( i \) is \( 4^i n/2^i \) and \( 3^i n/2^i \) respectively: number of children at depth \( i \) times the work at each child.

3. Total work is therefore \( n \sum_{i=0}^{L} 2^i \) and \( n \sum_{i=0}^{L} (3/2)^i \) respectively.
Recursion tree analysis

\[
\begin{align*}
\log_{2} n & \quad n \\
\frac{n}{2} & \quad \frac{n}{2} \\
\frac{n}{4} & \quad \frac{n}{4} \\
\frac{n}{8} & \quad \frac{n}{8} \\
\frac{n}{16} & \quad \frac{n}{16} \\
\vdots & \quad \vdots \\
\frac{3}{2} \cdot \frac{3}{2} & \quad \frac{3}{2} \cdot \frac{3}{2} \\
\end{align*}
\]

\[
\left(\frac{3}{2}\right)^d \cdot n = \left(\frac{3}{2}\right)^{\log_2 n} \cdot n = n
\]
Part II

Selecting in Unsorted Lists
### Rank of element in an array

**A**: an unsorted array of *n* integers

**Definition**

For $1 \leq j \leq n$, element of rank $j$ is the $j$'th smallest element in $A$.

<table>
<thead>
<tr>
<th>Unsorted array</th>
<th>16</th>
<th>14</th>
<th>34</th>
<th>20</th>
<th>12</th>
<th>5</th>
<th>3</th>
<th>19</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ranks</td>
<td>6</td>
<td>5</td>
<td>9</td>
<td>8</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td>7</td>
<td>3</td>
</tr>
<tr>
<td>Sort of array</td>
<td>3</td>
<td>5</td>
<td>11</td>
<td>12</td>
<td>14</td>
<td>16</td>
<td>19</td>
<td>20</td>
<td>34</td>
</tr>
</tbody>
</table>
Problem - Selection

Input  Unsorted array $A$ of $n$ integers \textbf{and} integer $j$

Goal  Find the $j$th smallest number in $A$ \textit{(rank $j$ number)}

Median: $j = \lfloor (n + 1) / 2 \rfloor$
Problem - Selection

Input  Unsorted array $A$ of $n$ integers and integer $j$

Goal  Find the $j$th smallest number in $A$ (rank $j$ number)

Median:  $j = \lfloor (n + 1)/2 \rfloor$

Simplifying assumption for sake of notation: elements of $A$ are distinct
Algorithm 1

1. Sort the elements in $A$
2. Pick $j$th element in sorted order

Time taken = $O(n \log n)$
Algorithm I

1. Sort the elements in $A$
2. Pick $j$th element in sorted order

Time taken $= O(n \log n)$

Do we need to sort? Is there an $O(n)$ time algorithm?
Algorithm II

If $j$ is small or $n - j$ is small then

1. Find $j$ smallest/largest elements in $A$ in $O(jn)$ time. (How?)
2. Time to find median is $O(n^2)$. 
Divide and Conquer Approach

1. Pick a pivot element \( a \) from \( A \)

2. Partition \( A \) based on \( a \).
   \[ A_{\text{less}} = \{x \in A \mid x \leq a\} \text{ and } A_{\text{greater}} = \{x \in A \mid x > a\} \]

3. \(|A_{\text{less}}| = j\): return \( a \)

4. \(|A_{\text{less}}| > j\): recursively find \( j \)th smallest element in \( A_{\text{less}} \)

5. \(|A_{\text{less}}| < j\): recursively find \( k \)th smallest element in \( A_{\text{greater}} \)
   where \( k = j - |A_{\text{less}}| \).
Example

\[
\begin{array}{c|c|c|c|c|c|c|c|c}
16 & 14 & 34 & 20 & 12 & 5 & 3 & 19 & 11 \\
\end{array}
\]

\[J = 468\]

14, 5, 3, 12, 11 \uparrow 16 20 34 19

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Time Analysis

1. Partitioning step: $O(n)$ time to scan $A$
2. How do we choose pivot? Recursive running time?
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Suppose we always choose pivot to be $A[1]$. 
Time Analysis

1. Partitioning step: \( O(n) \) time to scan \( A \)
2. How do we choose pivot? Recursive running time?

Suppose we always choose pivot to be \( A[1] \).

Say \( A \) is sorted in increasing order and \( j = n \).
Exercise: show that algorithm takes \( \Omega(n^2) \) time
A Better Pivot

Suppose pivot is the $\ell$th smallest element where $n/4 \leq \ell \leq 3n/4$. That is pivot is *approximately* in the middle of $A$.

Then $n/4 \leq |A_{\text{less}}| \leq 3n/4$ and $n/4 \leq |A_{\text{greater}}| \leq 3n/4$. If we apply recursion,

$$T(n) \leq n + T\left(\frac{3n}{4}\right)$$

Analysis a little bit later.

How do we find such a pivot?

Randomly?

In fact works!

Chandra & Manoj (UIUC)
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$$T(n) \leq T(3n/4) + O(n)$$

Implies $T(n) = O(n)$!
A Better Pivot

Suppose pivot is the $\ell$th smallest element where $\frac{n}{4} \leq \ell \leq \frac{3n}{4}$. That is pivot is approximately in the middle of $A$

Then $\frac{n}{4} \leq |A_{\text{less}}| \leq \frac{3n}{4}$ and $\frac{n}{4} \leq |A_{\text{greater}}| \leq \frac{3n}{4}$. If we apply recursion,

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How do we find such a pivot? Randomly? In fact works! Analysis a little bit later.
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Then $\frac{n}{4} \leq |A_{\text{less}}| \leq \frac{3n}{4}$ and $\frac{n}{4} \leq |A_{\text{greater}}| \leq \frac{3n}{4}$. If we apply recursion,

$$T(n) \leq T\left(\frac{3n}{4}\right) + O(n)$$

Implies $T(n) = O(n)$!

How do we find such a pivot? Randomly? In fact works! Analysis a little bit later.

Can we choose pivot deterministically?
Divide and Conquer Approach

A game of medians

**Idea**

1. Break input $A$ into many subarrays: $L_1, \ldots, L_k$.
2. Find median $m_i$ in each subarray $L_i$.
3. Find the median $x$ of the medians $m_1, \ldots, m_k$.
4. Intuition: The median $x$ should be close to being a good median of all the numbers in $A$.
5. Use $x$ as pivot in previous algorithm.
Example

| 11 | 7  | 3  | 42 | 174 | 310 | 1  | 92 | 87 | 12 | 19 | 15 |

\[ \frac{n}{3} \]

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Choosing the pivot
A clash of medians

1. Partition array $A$ into $\lceil n/5 \rceil$ lists of 5 items each.
   
   $L_1 = \{A[1], A[2], \ldots, A[5]\}$, $L_2 = \{A[6], \ldots, A[10]\}$, \ldots,
   
   $L_i = \{A[5i + 1], \ldots, A[5i - 4]\}$, \ldots,
   
   $L_{\lceil n/5 \rceil} = \{A[5 \lceil n/5 \rceil - 4], \ldots, A[n]\}$.

2. For each $i$ find median $b_i$ of $L_i$ using brute-force in $O(1)$ time.
   Total $O(n)$ time

3. Let $B = \{b_1, b_2, \ldots, b_{\lceil n/5 \rceil}\}$

4. Find median $b$ of $B$
Choosing the pivot

A clash of medians

1. Partition array $A$ into $\lceil n/5 \rceil$ lists of 5 items each.
   
   $L_1 = \{A[1], A[2], \ldots, A[5]\}$, $L_2 = \{A[6], \ldots, A[10]\}$, \ldots,
   
   $L_i = \{A[5i + 1], \ldots, A[5i - 4]\}$, \ldots,
   
   $L_{\lceil n/5 \rceil} = \{A[5\lceil n/5 \rceil - 4, \ldots, A[n]\}$.

2. For each $i$ find median $b_i$ of $L_i$ using brute-force in $O(1)$ time.
   Total $O(n)$ time

3. Let $B = \{b_1, b_2, \ldots, b_{\lceil n/5 \rceil}\}$

4. Find median $b$ of $B$

Lemma

Median of $B$ is an approximate median of $A$. That is, if $b$ is used a pivot to partition $A$, then $|A_{\text{less}}| \leq 7n/10 + 6$ and $|A_{\text{greater}}| \leq 7n/10 + 6$. 

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Algorithm for Selection

A storm of medians

**select**(A, j):

Form lists \(L_1, L_2, \ldots, L_{\lceil n/5 \rceil}\) where \(L_i = \{A[5i - 4], \ldots, A[5i]\}\)

Find median \(b_i\) of each \(L_i\) using brute-force

Find median \(b\) of \(B = \{b_1, b_2, \ldots, b_{\lceil n/5 \rceil}\}\)

Partition A into \(A_{\text{less}}\) and \(A_{\text{greater}}\) using \(b\) as pivot

if \(|A_{\text{less}}| = j\) return \(b\)

else if \(|A_{\text{less}}| > j\)

    return select\((A_{\text{less}}, j)\)

else

    return select\((A_{\text{greater}}, j - |A_{\text{less}}|)\)
Algorithm for Selection

A storm of medians

select(A, j):
Form lists \( L_1, L_2, \ldots, L_{\lceil n/5 \rceil} \) where \( L_i = \{A[5i-4], \ldots, A[5i]\} \)
Find median \( b_i \) of each \( L_i \) using brute-force
Find median \( b \) of \( B = \{b_1, b_2, \ldots, b_{\lceil n/5 \rceil}\} \)
Partition \( A \) into \( A_{less} \) and \( A_{greater} \) using \( b \) as pivot
if \( |A_{less}| = j \) return \( b \)
else if \( |A_{less}| > j \)
    return select\( (A_{less}, j) \)
else
    return select\( (A_{greater}, j - |A_{less}|) \)

How do we find median of \( B \)?
Algorithm for Selection

A storm of medians

**select**(*A*, *j*):

- Form lists $L_1, L_2, \ldots, L_{\lceil n/5 \rceil}$ where $L_i = \{A[5i - 4], \ldots, A[5i]\}$
- Find median $b_i$ of each $L_i$ using brute-force
- Find median $b$ of $B = \{b_1, b_2, \ldots, b_{\lceil n/5 \rceil}\}$
- Partition *A* into $A_{\text{less}}$ and $A_{\text{greater}}$ using $b$ as pivot
  - if ($|A_{\text{less}}|$) = *j* return *b*
  - else if ($|A_{\text{less}}|$) > *j*)
    - return **select**(A_{\text{less}}, *j*)
  - else
    - return **select**(A_{\text{greater}}, *j* − |A_{\text{less}}|)

How do we find median of *B*? Recursively!
**Algorithm for Selection**

**A storm of medians**

\[
\text{select}(A, j): \\
\text{Form lists } L_1, L_2, \ldots, L_{\lceil n/5 \rceil} \text{ where } L_i = \{A[5i-4], \ldots, A[5i]\} \\
\text{Find median } b_i \text{ of each } L_i \text{ using brute-force} \\
B = [b_1, b_2, \ldots, b_{\lceil n/5 \rceil}] \\
b = \text{select}(B, \lceil n/10 \rceil) \\
\text{Partition } A \text{ into } A_{\text{less}} \text{ and } A_{\text{greater}} \text{ using } b \text{ as pivot} \\
\text{if } (|A_{\text{less}}|) = j \text{ return } b \\
\text{else if } (|A_{\text{less}}|) > j \\
\quad \text{return select}(A_{\text{less}}, j) \\
\text{else} \\
\quad \text{return select}(A_{\text{greater}}, j - |A_{\text{less}}|)
\]
Running time of deterministic median selection

A dance with recurrences

\[ T(n) = T(\lceil n/5 \rceil) + \max \{ T(|A_{\text{less}}|), T(|A_{\text{greater}}|) \} + O(n) \]
Running time of deterministic median selection
A dance with recurrences

\[ T(n) = T\left(\lceil n/5 \rceil \right) + \max\{T(|A_{\text{less}}|), T(|A_{\text{greater}}|)\} + O(n) \]

From Lemma,

\[ T(n) \leq T\left(\lceil n/5 \rceil \right) + T\left(\lfloor 7n/10 + 6 \rfloor \right) + O(n) \]

and

\[ T(n) = O(1) \quad n < 10 \]
Running time of deterministic median selection

A dance with recurrences

\[ T(n) = T(\lceil n/5 \rceil) + \max\{T(|A_{\text{less}}|), T(|A_{\text{greater}}|)\} + O(n) \]

From Lemma,

\[ T(n) \leq T(\lceil n/5 \rceil) + T(\lfloor 7n/10 + 6 \rfloor) + O(n) \]

and

\[ T(n) = O(1) \quad n < 10 \]

Exercise: show that \( T(n) = O(n) \)
Median of Medians: Proof of Lemma

**Proposition**

There are at least \( \frac{3n}{10} - 6 \) elements greater than the median of medians \( b \).

**Figure**: Shaded elements are all greater than \( b \).
Median of Medians: Proof of Lemma

Figure: Shaded elements are all greater than $b$

Proposition

There are at least $\frac{3n}{10} - 6$ elements greater than the median of medians $b$.

Proof.

At least half of the $\lceil \frac{n}{5} \rceil$ groups have at least 3 elements larger than $b$, except for last group and the group containing $b$. Hence number of elements greater than $b$ is:

$$3\left(\lceil \left(\frac{1}{2}\right) \lceil \frac{n}{5} \rceil \rceil - 2 \right) \geq \frac{3n}{10} - 6$$
Proposition

There are at least \( \frac{3n}{10} - 6 \) elements greater than the median of medians \( b \).

Corollary

\[ |A_{\text{less}}| \leq \frac{7n}{10} + 6. \]

Via symmetric argument,

Corollary

\[ |A_{\text{greater}}| \leq \frac{7n}{10} + 6. \]
Questions to ponder

1. Why did we choose lists of size 5? Will lists of size 3 work?
2. Write a recurrence to analyze the algorithm’s running time if we choose a list of size $k$.

$$T(n) = T\left(\frac{9n}{10}\right) + n$$
Median of Medians Algorithm

Due to:
“Time bounds for selection”.
Median of Medians Algorithm

Due to:
“Time bounds for selection”.

How many Turing Award winners in the author list?
Median of Medians Algorithm

Due to:
“Time bounds for selection”.

How many Turing Award winners in the author list?
All except Vaughn Pratt!
Takeaway Points

1. Recursion tree method and guess and verify are the most reliable methods to analyze recursions in algorithms.
2. Recursive algorithms naturally lead to recurrences.
3. Sometimes one can look for certain type of recursive algorithms (reverse engineering) by understanding recurrences and their behavior.