Proving that a problem $X$ is NP-hard requires several steps:

- Choose a problem $Y$ that you already know is NP-hard.

- Describe an algorithm to solve $Y$, using an algorithm for $X$ as a subroutine. Typically this algorithm has the following form: Given an instance of $Y$, transform it into an instance of $X$, and then call the magic black-box algorithm for $X$.

- Prove that your algorithm is correct. This almost always requires two separate steps:
  - Prove that your algorithm transforms “yes” instances of $Y$ into “yes” instances of $X$.
  - Prove that your algorithm transforms “no” instances of $Y$ into “no” instances of $X$. Equivalently: Prove that if your transformation produces a “yes” instance of $X$, then it was given a “yes” instance of $Y$.

- Argue that your algorithm for $Y$ runs in polynomial time.

Proving that $X$ is NP-Complete requires you to additionally prove that $X \in NP$ by describing a non-deterministic polynomial-time algorithm for $X$. Typically this is not hard for the problems we consider but it is not always obvious.

1. Recall the following $k$COLOR problem: Given an undirected graph $G$, can its vertices be colored with $k$ colors, so that every edge touches vertices with two different colors?

   (a) Describe a direct polynomial-time reduction from 3COLOR to 4COLOR. Hint: Your reduction will take a graph $G$ and output another graph $G'$ such that $G'$ is 4-colorable if and only if $G$ is 3-colorable. You should think how an explicit 4-coloring for $G'$ would enable you to obtain an explicit 3-coloring for $G$.

   (b) Prove that $k$COLOR problem is NP-hard for any $k \geq 3$.

2. Describe a polynomial-time reduction from 3COLOR to SAT. Can you generalize it to reduce $k$COLOR to SAT. Hint: Use a variable $x(v, i)$ to indicate that $v$ is colored $i$ and express the constraints using clauses in CNF form.

3. Let $G = (V, E)$ be a directed graph with edge lengths $\ell(e), e \in E$. The lengths can be positive or negative. The Zero-Length-Cycle Problem is to decide whether $G$ has a cycle $C$ of length exactly equal to 0. Prove that this problem is NP-Complete. Hint: reduce Hamiltonian Path to Zero-Length-Cycle