# Problem Set 9
## CS 373: Theory of Computation

Assigned: April 18, 2013 Due on: April 25, 2013

**Instructions:** This homework has 3 problems that can be solved in groups of size at most 3. Please follow the homework guidelines given on the class website; submitions not following these guidelines will not be graded.

**Recommended Reading:** Lecture 23 and 24.

**Problem 1**. [Category: Comprehension+Proof] The Post Correspondence Problem (PCP) is the following. Given a set of tiles with two strings, one on the top and the other at the bottom, you want to determine if there is a list of these tiles (repetitions allowed) so that the string obtained by reading the top symbols is the same as the string obtained by reading the bottom symbols. This list is called a "match". For example, consider the set of tiles

$$\left\{ \left[\frac{b}{ca}\right], \left[\frac{a}{ab}\right], \left[\frac{ca}{a}\right], \left[\frac{abc}{c}\right] \right\}$$

If we consider the sequence of tiles

$$\left[\frac{a}{ab}\right] \left[\frac{b}{ca}\right] \left[\frac{ca}{a}\right] \left[\frac{a}{ab}\right] \left[\frac{abc}{c}\right]$$

the top string is $a \cdot b \cdot ca \cdot a \cdot abc = abcaaabc$ while the bottom string is $ab \cdot ca \cdot a \cdot ab \cdot c = abcaaabc$, is the same. However, not all sets of tiles have a match. For example,

$$\left\{ \left[\frac{abc}{a}\right], \left[\frac{ca}{a}\right], \left[\frac{acc}{ba}\right] \right\}$$

does not have a match. More formally, given

$$P = \left\{ \left[\frac{t_1}{b_1}\right], \left[\frac{t_2}{b_2}\right], \dots \left[\frac{t_k}{b_k}\right] \right\}$$

we need to determine if there is a sequence $i_1, i_2, \dots i_n$, where every $i_j \in \{1, 2, \dots k\}$, such that $t_{i_1} t_{i_2} \cdots t_{i_n} = b_{i_1} b_{i_2} \cdots b_{i_n}$. Thus,

$$\text{PCP} = \{\langle P \rangle \mid P \text{ is a set of tiles that has a match}\}$$

The PCP problem is known to be undecidable; interested students can read section 5.2 of Sipser's book.

Consider $\text{AMBIG}_{\text{CFG}} = \{\langle G \rangle \mid G \text{ is an ambiguous CFG}\}$. Prove that $\text{AMBIG}_{\text{CFG}}$ is undecidable by reducing PCP to $\text{AMBIG}_{\text{CFG}}$. *Hint:* Given an instance of PCP

$$P = \left\{ \left[\frac{t_1}{b_1}\right], \left[\frac{t_2}{b_2}\right], \dots \left[\frac{t_k}{b_k}\right] \right\}$$

construct a CFG $G$ with rules

$$\begin{aligned}
&S \to T \mid B \\
&T \to t_1 T a_1 \mid \cdots \mid t_k T a_k \mid t_1 a_1 \mid \cdots \mid t_k a_k \\
&B \to b_1 B a_1 \mid \cdots \mid b_k B a_k \mid b_1 a_1 \mid \cdots \mid b_k a_k
\end{aligned}$$

where $a_1, \dots a_k$ are new terminal symbols. Prove that this reduction is correct. **[10 points]**

**Problem 2**. [Category: Proof] Let $A, B \subseteq \{0,1\}^*$ be r.e. languages such that $A \cup B = \{0,1\}^*$ and $A \cap B \neq \emptyset$. Prove that $A \leq_m (A \cap B)$. **[10 points]**

**Problem 3**. [Category: Proof] Prove that a language $A$ is decidable iff $A \leq_m \mathbf{L}(0^*1^*)$. **[10 points]**