CS 273: Intro to Theory of Computation, Spring 2008
Problem Set 14
Due Tuesday, April 29nd, 10am

This homework contains three problems. Please submit each on a **separate sheet of paper**. Turn in your homework at Elaine Wilson's office (3229 Siebel).

1. Dovetailing

   (a) Briefly sketch an algorithm for enumerating all Turing machine encodings. Remember that each encoding is just a string, with some specific internal syntax (e.g. number of states, then number of symbols in Σ etc).

   (b) Now consider a language L that contains Turing machines which take other Turing machines as input. Specifically,

   $$L = \left\{ \langle M \rangle \;\middle|\; M \text{ halts on some input } \langle N \rangle \text{ where } N \text{ is a } \mathsf{TM} \right\}.$$

   As concrete examples of words in this language, consider the following TM M_1 and M_2.

   If M_1 is a decider that checks if a given input $\langle X \rangle$ (that encodes a TM) halts in ≤ 37 steps, then $\langle M_1 \rangle$ is in L.

   Suppose that M_2 halts and rejects if its input $\langle X \rangle$ is not the encoding of a TM, and M_2 spins off into an infinite loop if $\langle X \rangle$ is a TM encoding. Then, M_3 is not in L.

   Show that L is (nevertheless) TM recognizable by giving a recognizer for it.

2. Language classification revisited.

   Suppose that we have a set of Turing machine encodings defined by each of the following properties. That is, we have a set

   $$L = \left\{ \langle M \rangle \;\middle|\; M \text{ is a } \mathsf{TM} \text{ and } M \text{ has property } P \right\},$$

   and we are considering different ways to fill in P. Assume that the Turing machines M have only a single tape.

   (a) P is "M accepts some word $w \in \Sigma^*$ which has $|w| \leq 58$".

   (b) P is "M does not accept any word $w \in \Sigma^*$ which has $|w| \leq 249$".

   (c) P is "M stops on some string w containing the character $\mathtt{a}$ in ≤ 37 steps."

   (d) P is "M stops on some string $w \in \left\{ \mathtt{a}^n \mathtt{b}^n \;\middle|\; n \geq 0 \right\}$".

   (e) Given some additional (fixed) TM M', the property P is "there is a word w such that both M and M' accepts it."

   For each of these languages, determine whether it is Turing decidable, Turing recognizable, or not Turing recognizable. Briefly justify your answers.

3. LBAs emptiness revisited. (15 points)

Consider a TM M and a string w. Suppose that \$ and c are two fixed haracters that are not in M's tape alphabet Γ. Now define the following language

$$L_{M,w} = \left\{ z = w\$\$\$c^i \;\middle|\; i \geq 1 \text{ and } M \text{ accepts } w \text{ in at most } i \text{ steps} \right\}.$$

(a) Show that given M and w, the language $L_{M,w}$ can be decided by an LBA. That is, explain how to build a decider $D_{M,w}$ for $L_{M,w}$ that uses only the tape region where the input string z is written, and no additional space on the tape.

(b) M accepts w if and only if $L(D_{M,w}) \neq \emptyset$. Explain briefly why this is true.

(c) Assume that we can figure out how to compute the encoding $\langle D_{M,w} \rangle$, given $\langle M \rangle$ and w. Prove that the language

$$E_{\mathsf{LBA}} = \left\{ \langle M \rangle \;\middle|\; M \text{ is a } \mathsf{LBA} \text{ and } L(M) = \emptyset \right\}.$$

is undecidable, using (a) and (b).